

# GEOMETRIC CORRECTION FOR DIFFUSIVE EXPANSION OF STEADY NEUTRON TRANSPORT EQUATION

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**ABSTRACT.** We revisit the diffusive limit of a steady neutron transport equation in a 2-D unit disk  $\Omega = \{\vec{x} = (x_1, x_2) : |\vec{x}| \leq 1\}$  with one-speed velocity  $\Sigma = \{\vec{w} = (w_1, w_2) : \vec{w} \in \mathcal{S}^1\}$  as

$$(1) \quad \begin{cases} \epsilon \vec{w} \cdot \nabla_{\vec{x}} u^\epsilon + u^\epsilon - \bar{u}^\epsilon &= 0 & \text{in } \Omega, \\ u^\epsilon(\vec{x}_0, \vec{w}) &= g(\vec{x}_0, \vec{w}) & \text{for } \vec{w} \cdot \vec{n} < 0 \text{ and } \vec{x}_0 \in \partial\Omega, \end{cases}$$

where

$$(2) \quad \bar{u}^\epsilon(\vec{x}) = \frac{1}{2\pi} \int_{\mathcal{S}^1} u^\epsilon(\vec{x}, \vec{w}) d\vec{w}.$$

and  $\vec{n}$  is the outward normal vector on  $\partial\Omega$ , with the Knudsen number  $0 < \epsilon \ll 1$ . A classical result in [4] states that

$$(3) \quad \|u^\epsilon - U_0 - \mathcal{U}_0\|_{L^\infty} = O(\epsilon)$$

where  $\mathcal{U}_0$  is the Knudsen layer solution to the Milne problem (1.28) while  $U_0$  is the corresponding interior solution to the Laplace equation (1.29). We observe that the construction of the first order Knudsen layer fails in [4], due to the intrinsic singularity in the Milne problem. Instead, we are able to establish

$$(4) \quad \|u^\epsilon - U_0^\epsilon - \mathcal{U}_0^\epsilon\|_{L^\infty} = O(\epsilon)$$

where  $\mathcal{U}_0^\epsilon$  is the solution to the  $\epsilon$ -Milne problem (1.53) while  $U_0^\epsilon$  is the corresponding interior solution to the Laplace equation (1.54). Consequently, we deduce that

$$(5) \quad \|u^\epsilon - U_0 - \mathcal{U}_0\|_{L^\infty} = O(1)$$

for some data, and the classical Knudsen layer theory (3) is invalid in  $L^\infty$ .

**Keywords:**  $\epsilon$ -Milne problem, Knudsen layer solution, Geometric correction.

## 1. INTRODUCTION AND NOTATION

The diffusive limit  $\epsilon \rightarrow 0$  of the neutron transport equation (1) is a classical problem in kinetic theory. In the domain  $\Omega \times \Sigma$ , the neutron density  $u^\epsilon(\vec{x}, \vec{w})$  satisfies the equation (1).

Based on the flow direction, we can divide the boundary  $\Gamma = \{(\vec{x}, \vec{w}) : \vec{x} \in \partial\Omega\}$  into the in-flow boundary  $\Gamma^-$ , the out-flow boundary  $\Gamma^+$ , and the grazing set  $\Gamma^0$  as

$$(1.1) \quad \Gamma^- = \{(\vec{x}, \vec{w}) : \vec{x} \in \partial\Omega, \vec{w} \cdot \vec{n} < 0\},$$

$$(1.2) \quad \Gamma^+ = \{(\vec{x}, \vec{w}) : \vec{x} \in \partial\Omega, \vec{w} \cdot \vec{n} > 0\},$$

$$(1.3) \quad \Gamma^0 = \{(\vec{x}, \vec{w}) : \vec{x} \in \partial\Omega, \vec{w} \cdot \vec{n} = 0\}.$$

It is easy to see  $\Gamma = \Gamma^+ \cup \Gamma^- \cup \Gamma^0$ . Hence, the boundary condition is only given on  $\Gamma^-$ .

**1.1. Interior Expansion.** We define the interior expansion as follows:

$$(1.4) \quad U(\vec{x}, \vec{w}) \sim \sum_{k=0}^{\infty} \epsilon^k U_k(\vec{x}, \vec{w}),$$

where  $U_k^\epsilon$  can be defined by comparing the order of  $\epsilon$  via plugging (1.4) into the equation (1). Thus, we have

$$(1.5) \quad U_0 - \bar{U}_0 = 0,$$

$$(1.6) \quad U_1 - \bar{U}_1 = -\vec{w} \cdot \nabla_x U_0,$$

$$(1.7) \quad U_2 - \bar{U}_2 = -\vec{w} \cdot \nabla_x U_1,$$

$$(1.8) \quad \begin{aligned} & \dots \\ U_k - \bar{U}_k &= -\vec{w} \cdot \nabla_x U_{k-1}. \end{aligned}$$

The following analysis reveals the equation satisfied by  $U_k$ :

Plugging (1.5) into (1.6), we obtain

$$(1.9) \quad U_1 = \bar{U}_1 - \vec{w} \cdot \nabla_x \bar{U}_0.$$

Plugging (1.9) into (1.7), we get

$$(1.10) \quad U_2 - \bar{U}_2 = -\vec{w} \cdot \nabla_x (\bar{U}_1 - \vec{w} \cdot \nabla_x \bar{U}_0) = -\vec{w} \cdot \nabla_x \bar{U}_1 + \vec{w}^2 \Delta_x \bar{U}_0 + 2w_1 w_2 \partial_{x_1 x_2} \bar{U}_0.$$

Integrating (1.10) over  $\vec{w} \in \mathcal{S}^1$ , we achieve the final form

$$(1.11) \quad \Delta_x \bar{U}_0 = 0,$$

which further implies  $U_0(\vec{x}, \vec{w})$  satisfies the equation

$$(1.12) \quad \begin{cases} U_0 &= \bar{U}_0 \\ \Delta_x U_0 &= 0 \end{cases}$$

Similarly, we can derive  $U_k(\vec{x}, \vec{w})$  for  $k \geq 1$  satisfies

$$(1.13) \quad \begin{cases} U_k &= \bar{U}_k - \vec{w} \cdot \nabla_x U_{k-1} \\ \Delta_x \bar{U}_k &= 0 \end{cases}$$

**1.2. Milne Expansion.** In order to determine the boundary condition for  $U_k$ , it is well-known that we need to define the boundary layer expansion. Hence, we need several substitutions:

Substitution 1:

We consider the substitution into quasi-polar coordinates  $u^\epsilon(x_1, x_2, w_1, w_2) \rightarrow u^\epsilon(\mu, \theta, w_1, w_2)$  with  $(\mu, \theta, w_1, w_2) \in [0, 1] \times [-\pi, \pi] \times \mathcal{S}^1$  defined as

$$(1.14) \quad \begin{cases} x_1 &= (1 - \mu) \cos \theta, \\ x_2 &= (1 - \mu) \sin \theta, \\ w_1 &= w_1, \\ w_2 &= w_2. \end{cases}$$

Here  $\mu$  denotes the distance to the boundary  $\partial\Omega$  and  $\theta$  is the space angular variable. In these new variables, equation (1) can be rewritten as

$$(1.15) \quad \begin{cases} -\epsilon \left( w_1 \cos \theta + w_2 \sin \theta \right) \frac{\partial u^\epsilon}{\partial \mu} - \frac{\epsilon}{1 - \mu} \left( w_1 \sin \theta - w_2 \cos \theta \right) \frac{\partial u^\epsilon}{\partial \theta} + u^\epsilon - \frac{1}{2\pi} \int_{\mathcal{S}^1} u^\epsilon d\vec{w} = 0, \\ u^\epsilon(0, \theta, w_1, w_2) = g(\theta, w_1, w_2) \text{ for } w_1 \cos \theta + w_2 \sin \theta < 0. \end{cases}$$

Substitution 2:

We further define the stretched variable  $\eta$  by making the scaling transform for  $u^\epsilon(\mu, \theta, w_1, w_2) \rightarrow u^\epsilon(\eta, \theta, w_1, w_2)$  with  $(\eta, \theta, w_1, w_2) \in [0, 1/\epsilon] \times [-\pi, \pi] \times \mathcal{S}^1$  as

$$(1.16) \quad \begin{cases} \eta &= \mu/\epsilon, \\ \theta &= \theta, \\ w_1 &= w_1, \\ w_2 &= w_2, \end{cases}$$

which implies

$$(1.17) \quad \frac{\partial u^\epsilon}{\partial \mu} = \frac{1}{\epsilon} \frac{\partial u^\epsilon}{\partial \eta}.$$

Then equation (1) is transformed into

$$(1.18) \begin{cases} -\left(w_1 \cos \theta + w_2 \sin \theta\right) \frac{\partial u^\epsilon}{\partial \eta} - \frac{\epsilon}{1 - \epsilon \eta} \left(w_1 \sin \theta - w_2 \cos \theta\right) \frac{\partial u^\epsilon}{\partial \theta} + u^\epsilon - \frac{1}{2\pi} \int_{\mathcal{S}^1} u^\epsilon d\vec{w} = 0, \\ u^\epsilon(0, \theta, w_1, w_2) = g(\theta, w_1, w_2) \quad \text{for } w_1 \cos \theta + w_2 \sin \theta < 0. \end{cases}$$

Substitution 3:

Define the velocity substitution for  $u^\epsilon(\eta, \theta, w_1, w_2) \rightarrow u^\epsilon(\eta, \theta, \xi)$  with  $(\eta, \theta, \xi) \in [0, 1/\epsilon] \times [-\pi, \pi] \times [-\pi, \pi]$  as

$$(1.19) \quad \begin{cases} \eta &= \eta, \\ \theta &= \theta, \\ w_1 &= -\sin \xi, \\ w_2 &= -\cos \xi. \end{cases}$$

Here  $\xi$  denotes the velocity angular variable. We have the succinct form for (1) as

$$(1.20) \quad \begin{cases} \sin(\theta + \xi) \frac{\partial u^\epsilon}{\partial \eta} - \frac{\epsilon}{1 - \epsilon \eta} \cos(\theta + \xi) \frac{\partial u^\epsilon}{\partial \theta} + u^\epsilon - \frac{1}{2\pi} \int_{-\pi}^{\pi} u^\epsilon d\xi = 0, \\ u^\epsilon(0, \theta, \xi) = g(\theta, \xi) \quad \text{for } \sin(\theta + \xi) > 0. \end{cases}$$

We now define the Milne expansion of boundary layer as follows:

$$(1.21) \quad \mathcal{U}(\eta, \theta, \phi) \sim \sum_{k=0}^{\infty} \epsilon^k \mathcal{U}_k(\eta, \theta, \phi),$$

where  $\mathcal{U}_k$  can be determined by comparing the order of  $\epsilon$  via plugging (1.21) into the equation (1.20). Thus, in a neighborhood of the boundary, we have

$$(1.22) \quad \sin(\theta + \xi) \frac{\partial \mathcal{U}_0}{\partial \eta} + \mathcal{U}_0 - \bar{\mathcal{U}}_0 = 0,$$

$$(1.23) \quad \sin(\theta + \xi) \frac{\partial \mathcal{U}_1}{\partial \eta} + \mathcal{U}_1 - \bar{\mathcal{U}}_1 = \frac{1}{1 - \epsilon \eta} \cos(\theta + \xi) \frac{\partial \mathcal{U}_0}{\partial \theta},$$

$$(1.24) \quad \sin(\theta + \xi) \frac{\partial \mathcal{U}_k}{\partial \eta} + \mathcal{U}_k - \bar{\mathcal{U}}_k = \frac{1}{1 - \epsilon \eta} \cos(\theta + \xi) \frac{\partial \mathcal{U}_{k-1}}{\partial \theta},$$

where

$$(1.25) \quad \bar{\mathcal{U}}_k(\eta, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathcal{U}_k(\eta, \theta, \xi) d\xi.$$

The construction of  $U_k$  and  $\mathcal{U}_k$  in [4] can be summarized as follows:

Step 1: Construction of  $\mathcal{U}_0$  and  $U_0$ .

Assume the cut-off function  $\psi$  and  $\psi_0$  are defined as

$$(1.26) \quad \psi(\mu) = \begin{cases} 1 & 0 \leq \mu \leq 1/2, \\ 0 & 3/4 \leq \mu \leq \infty. \end{cases}$$

$$(1.27) \quad \psi_0(\mu) = \begin{cases} 1 & 0 \leq \mu \leq 1/4, \\ 0 & 3/8 \leq \mu \leq \infty. \end{cases}$$

Then the zeroth order boundary layer solution is defined as

$$(1.28) \quad \begin{cases} \mathcal{U}_0(\eta, \theta, \xi) &= \psi_0(\epsilon \eta) \left( f_0(\eta, \theta, \xi) - f_0(\infty, \theta) \right), \\ \sin(\theta + \xi) \frac{\partial f_0}{\partial \eta} + f_0 - \bar{f}_0 &= 0, \\ f_0(0, \theta, \xi) &= g(\theta, \xi) \quad \text{for } \sin(\theta + \xi) > 0, \\ \lim_{\eta \rightarrow \infty} f_0(\eta, \theta, \xi) &= f_0(\infty, \theta). \end{cases}$$

Assuming  $g \in L^\infty(\Gamma^-)$ , by Theorem 3.12, we can show there exists a unique solution  $f_0(\eta, \theta, \xi) \in L^\infty([0, \infty) \times [-\pi, \pi] \times [-\pi, \pi])$ . Hence,  $\mathcal{U}_0$  is well-defined. Then we can define the zeroth order interior solution as

$$(1.29) \quad \begin{cases} U_0 &= \bar{U}_0, \\ \Delta_x \bar{U}_0 &= 0 \text{ in } \Omega, \\ \bar{U}_0 &= f_0(\infty, \theta) \text{ on } \partial\Omega. \end{cases}$$

Step 2: Construction of  $\mathcal{U}_1$  and  $U_1$ .

Define the first order boundary layer solution as

$$(1.30) \quad \begin{cases} \mathcal{U}_1(\eta, \theta, \xi) &= \psi_0(\epsilon\eta) \left( f_1(\eta, \theta, \xi) - f_1(\infty, \theta) \right), \\ \sin(\theta + \xi) \frac{\partial f_1}{\partial \eta} + f_1 - \bar{f}_1 &= \cos(\theta + \xi) \frac{\psi(\epsilon\eta)}{1 - \epsilon\eta} \frac{\partial \mathcal{U}_0}{\partial \theta}, \\ f_1(0, \theta, \xi) &= \vec{w} \cdot \nabla_x U_0(\vec{x}_0, \vec{w}) \text{ for } \sin(\theta + \xi) > 0, \\ \lim_{\eta \rightarrow \infty} f_1(\eta, \theta, \xi) &= f_1(\infty, \theta). \end{cases}$$

where  $(\vec{x}_0, \vec{w})$  is the same point as  $(0, \theta, \xi)$ . Define the first order interior solution as

$$(1.31) \quad \begin{cases} U_1 &= \bar{U}_1 - \vec{w} \cdot \nabla_x U_0, \\ \Delta_x \bar{U}_1 &= 0 \text{ in } \Omega, \\ \bar{U}_1 &= f_1(\infty, \theta) \text{ on } \partial\Omega. \end{cases}$$

Step 3: Generalization to arbitrary  $k$ .

Similar to above procedure, we can define the  $k^{th}$  order boundary layer solution as

$$(1.32) \quad \begin{cases} \mathcal{U}_k(\eta, \theta, \xi) &= \psi_0(\epsilon\eta) \left( f_k(\eta, \theta, \xi) - f_k(\infty, \theta) \right), \\ \sin(\theta + \xi) \frac{\partial f_k}{\partial \eta} + f_k - \bar{f}_k &= \cos(\theta + \xi) \frac{\psi(\epsilon\eta)}{1 - \epsilon\eta} \frac{\partial \mathcal{U}_{k-1}}{\partial \theta}, \\ f_k(0, \theta, \xi) &= \vec{w} \cdot \nabla_x U_{k-1}(\vec{x}_0, \vec{w}) \text{ for } \sin(\theta + \xi) > 0, \\ \lim_{\eta \rightarrow \infty} f_k(\eta, \theta, \xi) &= f_k(\infty, \theta). \end{cases}$$

Define the  $k^{th}$  order interior solution as

$$(1.33) \quad \begin{cases} U_k &= \bar{U}_k - \vec{w} \cdot \nabla_x U_{k-1}, \\ \Delta_x \bar{U}_k &= 0 \text{ in } \Omega, \\ \bar{U}_k &= f_k(\infty, \theta) \text{ on } \partial\Omega. \end{cases}$$

In [4, pp.136], the author proved the following result:

**Theorem 1.1.** *Assume  $g(\vec{x}_0, \vec{w})$  is sufficiently smooth. Then for the steady neutron transport equation (1), the unique solution  $u^\epsilon(\vec{x}, \vec{w}) \in L^\infty(\Omega \times \mathcal{S}^1)$  satisfies*

$$(1.34) \quad \|u^\epsilon - U_0 - \mathcal{U}_0\|_{L^\infty} = O(\epsilon).$$

The goal of our paper is to reexamine the validity of Theorem 1.1. Our work begins with a crucial observation that based on Remark 3.15, the existence of solution  $f_1$  requires the source term

$$(1.35) \quad \cos(\theta + \xi) \frac{\psi}{1 - \epsilon\eta} \frac{\partial \mathcal{U}_0}{\partial \theta} \in L^\infty([0, \infty) \times [-\pi, \pi] \times [-\pi, \pi]).$$

Since the support of  $\psi(\epsilon\eta)$  depends on  $\epsilon$ , by (1.28), this in turn requires

$$(1.36) \quad \frac{\partial}{\partial \theta} \left( f_0(\eta, \theta, \xi) - f_0(\infty, \theta) \right) \in L^\infty([0, \infty) \times [-\pi, \pi] \times [-\pi, \pi])$$

Note that  $Z = \partial_\theta(f_0 - f_0(\infty, \theta))$  satisfies the equation

$$(1.37) \quad \begin{cases} \sin(\theta + \xi) \frac{\partial Z}{\partial \eta} + Z - \bar{Z} &= -\cos(\theta + \xi) \frac{\partial f_0}{\partial \eta}, \\ Z(0, \theta, \xi) &= \frac{\partial g(\theta, \xi)}{\partial \theta} - \frac{\partial f_0(\infty, \theta)}{\partial \theta} \text{ for } \sin(\theta + \xi) > 0, \\ \lim_{\eta \rightarrow \infty} Z(\eta, \theta, \xi) &= Z(\infty, \theta). \end{cases}$$

In order for  $Z \in L^\infty([0, \infty) \times [-\pi, \pi] \times [-\pi, \pi])$ , assuming the boundary data  $\partial_\theta g \in L^\infty(\Gamma^-)$ , we require the source term

$$(1.38) \quad -\cos(\theta + \xi) \frac{\partial f_0}{\partial \eta} \in L^\infty([0, \infty) \times [-\pi, \pi] \times [-\pi, \pi]).$$

On the other hand, as shown by Lemma A.1, we can show for specific  $g$ , it holds that  $\partial_\eta f_0 \notin L^\infty([0, \infty) \times [-\pi, \pi] \times [-\pi, \pi])$ . Due to intrinsic singularity for (1.28), the construction in [4] breaks down.

In fact, in general geometry with curved boundary, we need to control the normal derivative of the boundary layer solution for the Milne expansion.

**1.3.  $\epsilon$ -Milne Expansion with Geometric Correction.** Our main goal is to overcome the difficulty in estimating

$$(1.39) \quad \cos(\theta + \xi) \frac{\psi}{1 - \epsilon \eta} \frac{\partial \mathcal{U}_k}{\partial \theta}.$$

We introduce one more substitution to decompose the term (1.39).

Substitution 4:

We make the rotation substitution for  $u^\epsilon(\eta, \theta, \xi) \rightarrow u^\epsilon(\eta, \theta, \phi)$  with  $(\eta, \theta, \phi) \in [0, 1/\epsilon) \times [-\pi, \pi] \times [-\pi, \pi]$  as

$$(1.40) \quad \begin{cases} \eta &= \eta, \\ \theta &= \theta, \\ \phi &= \theta + \xi, \end{cases}$$

and transform the equation (1) into

$$(1.41) \quad \begin{cases} \sin \phi \frac{\partial u^\epsilon}{\partial \eta} - \frac{\epsilon}{1 - \epsilon \eta} \cos \phi \left( \frac{\partial u^\epsilon}{\partial \phi} + \frac{\partial u^\epsilon}{\partial \theta} \right) + u^\epsilon - \frac{1}{2\pi} \int_{-\pi}^{\pi} u^\epsilon d\phi = 0, \\ u^\epsilon(0, \theta, \phi) = g(\theta, \phi) \text{ for } \sin \phi > 0. \end{cases}$$

Inspired by [6], [12] and [1], the most important idea is to include the most singular term

$$(1.42) \quad -\frac{\epsilon}{1 - \epsilon \eta} \cos \phi \frac{\partial u^\epsilon}{\partial \phi}$$

in the Milne problem.

We define the  $\epsilon$ -Milne expansion with geometric correction of boundary layer as follows:

$$(1.43) \quad \mathcal{U}^\epsilon(\eta, \theta, \phi) \sim \sum_{k=0}^{\infty} \epsilon^k \mathcal{U}_k^\epsilon(\eta, \theta, \phi),$$

where  $\mathcal{U}_k^\epsilon$  can be determined by comparing the order of  $\epsilon$  via plugging (1.43) into the equation (1.41). Thus, in a neighborhood of the boundary, we have

$$(1.44) \quad \sin \phi \frac{\partial \mathcal{U}_0^\epsilon}{\partial \eta} + \frac{\epsilon}{1 - \epsilon \eta} \cos \phi \frac{\partial \mathcal{U}_0^\epsilon}{\partial \phi} + \mathcal{U}_0^\epsilon - \bar{\mathcal{U}}_0^\epsilon = 0,$$

$$(1.45) \quad \sin \phi \frac{\partial \mathcal{U}_1^\epsilon}{\partial \eta} + \frac{\epsilon}{1 - \epsilon \eta} \cos \phi \frac{\partial \mathcal{U}_1^\epsilon}{\partial \phi} + \mathcal{U}_1^\epsilon - \bar{\mathcal{U}}_1^\epsilon = \frac{1}{1 - \epsilon \eta} \cos \phi \frac{\partial \mathcal{U}_0^\epsilon}{\partial \theta},$$

$$(1.46) \quad \sin \phi \frac{\partial \mathcal{U}_k^\epsilon}{\partial \eta} + \frac{\epsilon}{1 - \epsilon \eta} \cos \phi \frac{\partial \mathcal{U}_k^\epsilon}{\partial \phi} + \mathcal{U}_k^\epsilon - \bar{\mathcal{U}}_k^\epsilon = \frac{1}{1 - \epsilon \eta} \cos \phi \frac{\partial \mathcal{U}_{k-1}^\epsilon}{\partial \theta}.$$

where

$$(1.47) \quad \bar{\mathcal{U}}_k^\epsilon(\eta, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathcal{U}_k^\epsilon(\eta, \theta, \phi) d\phi.$$

It is important to note the solution  $\mathcal{U}_k^\epsilon$  depends on  $\epsilon$ .

We refer to the cut-off function  $\psi$  and  $\psi_0$  as (1.26) and (1.27), and define the force as

$$(1.48) \quad F(\epsilon; \eta) = -\frac{\epsilon\psi(\epsilon\eta)}{1 - \epsilon\eta},$$

Define the interior expansion as follows:

$$(1.49) \quad U^\epsilon(\vec{x}, \vec{w}) \sim \sum_{k=0}^{\infty} \epsilon^k U_k^\epsilon(\vec{x}, \vec{w})$$

where  $U_k^\epsilon$  satisfies the same equations as  $U_k$  in (1.12) and (1.13). Here, to highlight its dependence on  $\epsilon$  via the  $\epsilon$ -Milne problem and boundary data, we add the superscript  $\epsilon$ .

The bridge between the interior solution and the boundary layer solution is the boundary condition of (1), so we first consider the boundary condition expansion

$$(1.50) \quad U_0^\epsilon + \mathcal{U}_0^\epsilon = g,$$

$$(1.51) \quad U_1^\epsilon + \mathcal{U}_1^\epsilon = 0,$$

$$(1.52) \quad \begin{aligned} &\dots \\ U_k^\epsilon + \mathcal{U}_k^\epsilon &= 0. \end{aligned}$$

The construction of  $U_k^\epsilon$  and  $\mathcal{U}_k^\epsilon$  are as follows:

Step 1: Construction of  $\mathcal{U}_0^\epsilon$  and  $U_0^\epsilon$ .

Define the zeroth order boundary layer solution as

$$(1.53) \quad \left\{ \begin{aligned} \mathcal{U}_0^\epsilon(\eta, \theta, \phi) &= \psi_0(\epsilon\eta) \left( f_0^\epsilon(\eta, \theta, \phi) - f_0^\epsilon(\infty, \theta) \right), \\ \sin \phi \frac{\partial f_0^\epsilon}{\partial \eta} + F(\epsilon; \eta) \cos \phi \frac{\partial f_0^\epsilon}{\partial \phi} + f_0^\epsilon - \bar{f}_0^\epsilon &= 0, \\ f_0^\epsilon(0, \theta, \phi) &= g(\theta, \phi) \text{ for } \sin \phi > 0, \\ \lim_{\eta \rightarrow \infty} f_0^\epsilon(\eta, \theta, \phi) &= f_0^\epsilon(\infty, \theta). \end{aligned} \right.$$

In contrast to the classical Milne problem (1.28), the key advantage is, due to the geometry,  $\frac{\partial F(\epsilon; \eta)}{\partial \theta} = 0$ , such that (1.53) is invariant in  $\theta$ .

Then we define the zeroth order interior solution  $U_0^\epsilon(\vec{x})$  as

$$(1.54) \quad \left\{ \begin{aligned} U_0^\epsilon &= \bar{U}_0^\epsilon, \\ \Delta_x \bar{U}_0^\epsilon &= 0 \text{ in } \Omega, \\ \bar{U}_0^\epsilon &= f_0^\epsilon(\infty, \theta) \text{ on } \partial\Omega. \end{aligned} \right.$$

Step 2: Estimates of  $\frac{\partial \mathcal{U}_0^\epsilon}{\partial \theta}$ .

By Theorem 3.13, we can easily see  $f_0^\epsilon$  is well-defined in  $L^\infty(\Omega \times \mathcal{S}^1)$  and approaches  $f_0^\epsilon(\infty)$  exponentially fast as  $\eta \rightarrow \infty$ . Then we can naturally derive  $Z = \partial_\theta(f_0^\epsilon - f_0^\epsilon(\infty))$  also satisfies the same type of  $\epsilon$ -Milne problem

$$(1.55) \quad \left\{ \begin{aligned} \sin \phi \frac{\partial Z}{\partial \eta} + F(\epsilon; \eta) \cos \phi \frac{\partial Z}{\partial \phi} + Z - \bar{Z} &= 0, \\ Z(0, \theta, \phi) &= \frac{\partial g}{\partial \theta} - \frac{\partial f_0^\epsilon(\infty)}{\partial \theta} \text{ for } \sin \phi > 0, \\ \lim_{\eta \rightarrow \infty} Z(\eta, \phi) &= C. \end{aligned} \right.$$

By Theorem 3.13, we can see  $Z \rightarrow C$  exponentially fast as  $\eta \rightarrow \infty$ . It is natural to obtain this constant  $C$  must be zero. Hence, if  $g \in C^r(\Gamma^-)$ , it is obvious to check  $f_0^\epsilon(\infty) \in C^r(\partial\Omega)$ . By the standard elliptic

estimate in (1.54), there exists a unique solution  $\bar{U}_0^\epsilon \in W^{r,p}(\Omega)$  for arbitrary  $p \geq 2$  satisfying

$$(1.56) \quad \|\bar{U}_0^\epsilon\|_{W^{r,p}(\Omega)} \leq C(\Omega) \|f_0^\epsilon(\infty)\|_{W^{r-1/p,p}(\partial\Omega)},$$

which implies  $\nabla_x \bar{U}_0^\epsilon \in W^{r-1,p}(\Omega)$ ,  $\nabla_x \bar{U}_0^\epsilon \in W^{r-1-1/p,p}(\partial\Omega)$  and  $\bar{U}_0^\epsilon \in C^{r-1,1-2/p}(\Omega)$ .

Step 3: Construction of  $\mathcal{U}_1^\epsilon$  and  $U_1^\epsilon$ .

Define the first order boundary layer solution as

$$(1.57) \quad \begin{cases} \mathcal{U}_1^\epsilon(\eta, \theta, \phi) &= \psi_0(\epsilon\eta) \left( f_1^\epsilon(\eta, \theta, \phi) - f_1^\epsilon(\infty, \theta) \right), \\ \sin \phi \frac{\partial f_1^\epsilon}{\partial \eta} + F(\epsilon; \eta) \cos \phi \frac{\partial f_1^\epsilon}{\partial \phi} + f_1^\epsilon - \bar{f}_1^\epsilon &= \frac{\psi(\epsilon\eta)}{1 - \epsilon\eta} \cos \phi \frac{\partial \mathcal{U}_0^\epsilon}{\partial \theta}, \\ f_1^\epsilon(0, \theta, \phi) &= \vec{w} \cdot \nabla_x U_0^\epsilon(\vec{x}_0, \vec{w}) \text{ for } \sin \phi > 0, \\ \lim_{\eta \rightarrow \infty} f_1^\epsilon(\eta, \theta, \phi) &= f_1^\epsilon(\infty, \theta). \end{cases}$$

where  $(\vec{x}_0, \vec{w})$  is the same point as  $(0, \theta, \phi)$ . Then we define the first order interior solution  $U_1^\epsilon(\vec{x})$  as

$$(1.58) \quad \begin{cases} U_1^\epsilon &= \bar{U}_1^\epsilon - \vec{w} \cdot \nabla_x U_0^\epsilon, \\ \Delta_x \bar{U}_1^\epsilon &= 0 \text{ in } \Omega, \\ \bar{U}_1^\epsilon &= f_1^\epsilon(\infty, \theta) \text{ on } \partial\Omega. \end{cases}$$

Step 4: Estimates of  $\frac{\partial \mathcal{U}_1^\epsilon}{\partial \theta}$ .

By Theorem 3.13, we can easily see  $f_1^\epsilon$  is well-defined in  $L^\infty(\Omega \times \mathcal{S}^1)$  and approaches  $f_1^\epsilon(\infty)$  exponentially fast as  $\eta \rightarrow \infty$ . Also, since  $\vec{w} \cdot \nabla_x U_0^\epsilon \in W^{r-1-1/p,p}(\partial\Omega)$ ,  $\partial_\theta f_1^\epsilon$  is well-defined and decays exponentially fast. Hence,  $f_1^\epsilon(\infty, \theta) \in W^{r-1-1/p,p}(\partial\Omega)$ . By the standard elliptic estimate in (1.58), there exists a unique solution  $\bar{U}_1^\epsilon \in W^{r-1,p}(\Omega)$  and satisfies

$$(1.59) \quad \|\bar{U}_1^\epsilon\|_{W^{r-1,p}(\Omega)} \leq C(\Omega) \|f_1^\epsilon(\infty)\|_{W^{r-1-1/p,p}(\partial\Omega)},$$

which implies  $\nabla_x \bar{U}_1^\epsilon \in W^{r-2,p}(\Omega)$ ,  $\nabla_x \bar{U}_1^\epsilon \in W^{r-2-1/p,p}(\partial\Omega)$  and  $\bar{U}_1^\epsilon \in C^{r-2,1-2/p}(\Omega)$ .

Step 5: Generalization to arbitrary  $k$ .

In a similar fashion, as long as  $g$  is sufficiently smooth, above process can go on. We construct the  $k^{th}$  order boundary layer solution as

$$(1.60) \quad \begin{cases} \mathcal{U}_k^\epsilon(\eta, \theta, \phi) &= \psi_0(\epsilon\eta) \left( f_k^\epsilon(\eta, \theta, \phi) - f_k^\epsilon(\infty, \theta) \right), \\ \sin \phi \frac{\partial f_k^\epsilon}{\partial \eta} + F(\epsilon; \eta) \cos \phi \frac{\partial f_k^\epsilon}{\partial \phi} + f_k^\epsilon - \bar{f}_k^\epsilon &= \frac{\psi(\epsilon\eta)}{1 - \epsilon\eta} \cos \phi \frac{\partial \mathcal{U}_{k-1}^\epsilon}{\partial \theta}, \\ f_k^\epsilon(0, \theta, \phi) &= \vec{w} \cdot \nabla_x U_{k-1}^\epsilon(\vec{x}_0, \vec{w}) \text{ for } \sin \phi > 0, \\ \lim_{\eta \rightarrow \infty} f_k^\epsilon(\eta, \theta, \phi) &= f_k^\epsilon(\infty, \theta). \end{cases}$$

Then we define the  $k^{th}$  order interior solution as

$$(1.61) \quad \begin{cases} U_k^\epsilon &= \bar{U}_k^\epsilon - \vec{w} \cdot \nabla_x U_{k-1}^\epsilon, \\ \Delta_x \bar{U}_k^\epsilon &= 0 \text{ in } \Omega, \\ \bar{U}_k^\epsilon &= f_k^\epsilon(\infty, \theta) \text{ on } \partial\Omega. \end{cases}$$

For  $g \in C^{k+1}(\Gamma^-)$ , the interior solution and boundary layer solution can be well-defined up to  $k^{th}$  order, i.e. up to  $U_k^\epsilon$  and  $\mathcal{U}_k^\epsilon$ .

#### 1.4. Main Results.

**Theorem 1.2.** Assume  $g(\vec{x}_0, \vec{w}) \in C^3(\Gamma^-)$ . Then for the steady neutron transport equation (1), the unique solution  $u^\epsilon(\vec{x}, \vec{w}) \in L^\infty(\Omega \times \mathcal{S}^1)$  satisfies

$$(1.62) \quad \|u^\epsilon - U_0^\epsilon - \mathcal{U}_0^\epsilon\|_{L^\infty} = O(\epsilon)$$

where  $U_0^\epsilon$  and  $\mathcal{U}_0^\epsilon$  are defined in (1.54) and (1.53). Moreover, if  $g(\theta, \phi) = \cos \phi$ , then there exists a  $C > 0$  such that

$$(1.63) \quad \|u^\epsilon - U_0 - \mathcal{U}_0\|_{L^\infty} \geq C > 0$$

when  $\epsilon$  is sufficiently small, where  $U_0$  and  $\mathcal{U}_0$  are defined in (1.29) and (1.28).

For the diffusive boundary case, the zeroth order classical Knudsen layer is always absent. Our method leads to the invalidity of the classical Knudsen layer expansion at first order. See Theorem 5.9.

Our results demonstrates that the classical Knudsen layer expansion in Theorem 1.1 breaks down in a unit disk in  $L^\infty$ . Even though the equation (1.53) only modifies the equation (1.28) with an  $\epsilon$  order term, the difference between the solutions can be order 1 via a contradiction argument in Step 5 of Section 4. We remark that  $L^\infty$  is a natural space to characterize boundary layer contributions, whose  $L^p$  norm is  $O(\epsilon^{1/p})$  for  $1 \leq p < \infty$ . Unfortunately, our new expansion can only be established in a disk, because the  $\theta$  derivative can be controlled due to the constant curvature along the boundary. However, a new mathematical theory is needed to characterize such a diffusive limit in a general domain. Our analysis is based on a careful study of the  $\epsilon$ -Milne problem with geometric correction. Our paper is self-contained and without any use of probabilistic techniques as in [4].

Throughout this paper,  $C > 0$  denotes a constant that only depends on the parameter  $\Omega$ , but does not depend on the data. It is referred as universal and can change from one inequality to another. When we write  $C(z)$ , it means a certain positive constant depending on the quantity  $z$ .

Our paper is organized as follows: in Section 2, we first establish the  $L^\infty$  well-posedness of the equation (1); in Section 3, we give a complete analysis of the  $\epsilon$ -Milne problem with geometric correction; in Section 4, we give the detailed proof of Theorem 1.2 and finally, in Section 5, we discuss the case of diffusive boundary.

## 2. WELL-POSEDNESS OF STEADY NEUTRON TRANSPORT EQUATION

In this section, we consider the well-posedness of the steady neutron transport equation

$$(2.1) \quad \begin{cases} \epsilon \vec{w} \cdot \nabla_x u + u - \bar{u} &= f(\vec{x}, \vec{w}) \text{ in } \Omega, \\ u(\vec{x}_0, \vec{w}) &= g(\vec{x}_0, \vec{w}) \text{ for } \vec{x}_0 \in \partial\Omega \text{ and } \vec{w} \cdot \vec{n} < 0. \end{cases}$$

We define the  $L^2$  and  $L^\infty$  norms in  $\Omega \times \mathcal{S}^1$  as usual:

$$(2.2) \quad \|f\|_{L^2(\Omega \times \mathcal{S}^1)} = \left( \int_{\Omega} \int_{\mathcal{S}^1} |f(\vec{x}, \vec{w})|^2 d\vec{w} d\vec{x} \right)^{1/2},$$

$$(2.3) \quad \|f\|_{L^\infty(\Omega \times \mathcal{S}^1)} = \sup_{(\vec{x}, \vec{w}) \in \Omega \times \mathcal{S}^1} |f(\vec{x}, \vec{w})|.$$

Define the  $L^2$  and  $L^\infty$  norms on the boundary as follows:

$$(2.4) \quad \|f\|_{L^2(\Gamma)} = \left( \iint_{\Gamma} |f(\vec{x}, \vec{w})|^2 |\vec{w} \cdot \vec{n}| d\vec{w} d\vec{x} \right)^{1/2},$$

$$(2.5) \quad \|f\|_{L^2(\Gamma^\pm)} = \left( \iint_{\Gamma^\pm} |f(\vec{x}, \vec{w})|^2 |\vec{w} \cdot \vec{n}| d\vec{w} d\vec{x} \right)^{1/2},$$

$$(2.6) \quad \|f\|_{L^\infty(\Gamma)} = \sup_{(\vec{x}, \vec{w}) \in \Gamma} |f(\vec{x}, \vec{w})|,$$

$$(2.7) \quad \|f\|_{L^\infty(\Gamma^\pm)} = \sup_{(\vec{x}, \vec{w}) \in \Gamma^\pm} |f(\vec{x}, \vec{w})|.$$

**2.1. Preliminaries.** In order to show the  $L^2$  and  $L^\infty$  well-posedness of the equation (2.1), we start with some preparations with the penalized neutron transport equation.

**Lemma 2.1.** *Assume  $f(\vec{x}, \vec{w}) \in L^\infty(\Omega \times \mathcal{S}^1)$  and  $g(x_0, \vec{w}) \in L^\infty(\Gamma^-)$ . Then for the penalized transport equation*

$$(2.8) \quad \begin{cases} \lambda u_\lambda + \epsilon \vec{w} \cdot \nabla_x u_\lambda + u_\lambda &= f(\vec{x}, \vec{w}) \text{ in } \Omega, \\ u_\lambda(\vec{x}_0, \vec{w}) &= g(\vec{x}_0, \vec{w}) \text{ for } \vec{x}_0 \in \partial\Omega \text{ and } \vec{w} \cdot \vec{n} < 0. \end{cases}$$

with  $\lambda > 0$  as a penalty parameter, there exists a solution  $u_\lambda(\vec{x}, \vec{w}) \in L^\infty(\Omega \times \mathcal{S}^1)$  satisfying

$$(2.9) \quad \|u_\lambda\|_{L^\infty(\Omega \times \mathcal{S}^1)} \leq \|f\|_{L^\infty(\Omega \times \mathcal{S}^1)} + \|g\|_{L^\infty(\Gamma^-)}.$$



*Proof.* The characteristics  $(X(s), W(s))$  of the equation (2.8) which goes through  $(\vec{x}, \vec{w})$  is defined by

$$(2.10) \quad \begin{cases} (X(0), W(0)) &= (\vec{x}, \vec{w}) \\ \frac{dX(s)}{ds} &= \epsilon W(s), \\ \frac{dW(s)}{ds} &= 0. \end{cases}$$

which implies

$$(2.11) \quad \begin{cases} X(s) &= \vec{x} + (\epsilon \vec{w})s \\ W(s) &= \vec{w} \end{cases}$$

Hence, we can rewrite the equation (2.8) along the characteristics as

$$(2.12) \quad u_\lambda(\vec{x}, \vec{w}) = g(\vec{x} - \epsilon t_b \vec{w}, \vec{w}) e^{-(1+\lambda)t_b} + \int_0^{t_b} f(\vec{x} - \epsilon(t_b - s)\vec{w}, \vec{w}) e^{-(1+\lambda)(t_b - s)} ds,$$

where the backward exit time  $t_b$  is defined as

$$(2.13) \quad t_b(\vec{x}, \vec{w}) = \inf\{t \geq 0 : (\vec{x} - \epsilon t \vec{w}, \vec{w}) \in \Gamma^-\}.$$

Then we can naturally estimate

$$(2.14) \quad \begin{aligned} \|u_\lambda\|_{L^\infty(\Omega \times \mathcal{S}^1)} &\leq e^{-(1+\lambda)t_b} \|g\|_{L^\infty(\Gamma^-)} + \frac{1 - e^{-(1+\lambda)t_b}}{1 + \lambda} \|f\|_{L^\infty(\Omega \times \mathcal{S}^1)} \\ &\leq \|g\|_{L^\infty(\Gamma^-)} + \|f\|_{L^\infty(\Omega \times \mathcal{S}^1)}. \end{aligned}$$

Since  $u_\lambda$  can be explicitly traced back to the boundary data, the existence naturally follows from above estimate.  $\square$

**Lemma 2.2.** Assume  $f(\vec{x}, \vec{w}) \in L^\infty(\Omega \times \mathcal{S}^1)$  and  $g(x_0, \vec{w}) \in L^\infty(\Gamma^-)$ . Then for the penalized neutron transport equation

$$(2.15) \quad \begin{cases} \lambda u_\lambda + \epsilon \vec{w} \cdot \nabla_x u_\lambda + u_\lambda - \bar{u}_\lambda &= f(\vec{x}, \vec{w}) \text{ in } \Omega, \\ u_\lambda(\vec{x}_0, \vec{w}) &= g(\vec{x}_0, \vec{w}) \text{ for } \vec{x}_0 \in \partial\Omega \text{ and } \vec{w} \cdot \vec{n} < 0. \end{cases}$$

with  $\lambda > 0$ , there exists a solution  $u_\lambda(\vec{x}, \vec{w}) \in L^\infty(\Omega \times \mathcal{S}^1)$  satisfying

$$(2.16) \quad \|u_\lambda\|_{L^\infty(\Omega \times \mathcal{S}^1)} \leq \frac{1 + \lambda}{\lambda} \left( \|f\|_{L^\infty(\Omega \times \mathcal{S}^1)} + \|g\|_{L^\infty(\Gamma^-)} \right).$$

*Proof.* We define an approximating sequence  $\{u_\lambda^k\}_{k=0}^\infty$ , where  $u_\lambda^0 = 0$  and

$$(2.17) \quad \begin{cases} \lambda u_\lambda^k + \epsilon \vec{w} \cdot \nabla_x u_\lambda^k + u_\lambda^k - \bar{u}_\lambda^{k-1} &= f(\vec{x}, \vec{w}) \text{ in } \Omega, \\ u_\lambda^k(\vec{x}_0, \vec{w}) &= g(\vec{x}_0, \vec{w}) \text{ for } \vec{x}_0 \in \partial\Omega \text{ and } \vec{w} \cdot \vec{n} < 0. \end{cases}$$

By Lemma 2.1, this sequence is well-defined and  $\|u_\lambda^k\|_{L^\infty(\Omega \times \mathcal{S}^1)} < \infty$ .

The characteristics and the backward exit time are defined as (2.10) and (2.13), so we rewrite equation (2.17) along the characteristics as

$$(2.18) \quad u_\lambda^k(\vec{x}, \vec{w}) = g(\vec{x} - \epsilon t_b \vec{w}, \vec{w}) e^{-(1+\lambda)t_b} + \int_0^{t_b} (f + \bar{u}_\lambda^{k-1})(\vec{x} - \epsilon(t_b - s)\vec{w}, \vec{w}) e^{-(1+\lambda)(t_b - s)} ds.$$

We define the difference  $v^k = u_\lambda^k - u_\lambda^{k-1}$  for  $k \geq 1$ . Recall (2) for  $\bar{v}^k$ , then  $v^k$  satisfies

$$v^{k+1}(\vec{x}, \vec{w}) = \int_0^{t_b} \bar{v}^k(\vec{x} - \epsilon(t_b - s)\vec{w}, \vec{w}) e^{-(1+\lambda)(t_b - s)} ds.$$

Since  $\|\bar{v}^k\|_{L^\infty(\Omega \times \mathcal{S}^1)} \leq \|v^k\|_{L^\infty(\Omega \times \mathcal{S}^1)}$ , we can directly estimate

$$(2.19) \quad \|v^{k+1}\|_{L^\infty(\Omega \times \mathcal{S}^1)} \leq \|v^k\|_{L^\infty(\Omega \times \mathcal{S}^1)} \int_0^{t_b} e^{-(1+\lambda)(t_b - s)} ds \leq \frac{1 - e^{-(1+\lambda)t_b}}{1 + \lambda} \|v^k\|_{L^\infty(\Omega \times \mathcal{S}^1)}.$$

Hence, we naturally have

$$(2.20) \quad \|v^{k+1}\|_{L^\infty(\Omega \times \mathcal{S}^1)} \leq \frac{1}{1 + \lambda} \|v^k\|_{L^\infty(\Omega \times \mathcal{S}^1)}.$$

Thus, this is a contraction sequence for  $\lambda > 0$ . Considering  $v^1 = u_\lambda^1$ , we have

$$(2.21) \quad \|v^k\|_{L^\infty(\Omega \times \mathcal{S}^1)} \leq \left(\frac{1}{1+\lambda}\right)^{k-1} \|u_\lambda^1\|_{L^\infty(\Omega \times \mathcal{S}^1)},$$

for  $k \geq 1$ . Therefore,  $u_\lambda^k$  converges strongly in  $L^\infty$  to a limit solution  $u_\lambda$  satisfying

$$(2.22) \quad \|u_\lambda\|_{L^\infty(\Omega \times \mathcal{S}^1)} \leq \sum_{k=1}^{\infty} \|v^k\|_{L^\infty(\Omega \times \mathcal{S}^1)} \leq \frac{1+\lambda}{\lambda} \|u_\lambda^1\|_{L^\infty(\Omega \times \mathcal{S}^1)}.$$

Since  $u_\lambda^1$  can be rewritten along the characteristics as

$$(2.23) \quad u_\lambda^1(\vec{x}, \vec{w}) = g(\vec{x} - \epsilon t_b \vec{w}, \vec{w}) e^{-(1+\lambda)t_b} + \int_0^{t_b} f(\vec{x} - \epsilon(t_b - s)\vec{w}, \vec{w}) e^{-(1+\lambda)(t_b-s)} ds,$$

based on Lemma 2.1, we can directly estimate

$$(2.24) \quad \|u_\lambda^1\|_{L^\infty(\Omega \times \mathcal{S}^1)} \leq \|f\|_{L^\infty(\Omega \times \mathcal{S}^1)} + \|g\|_{L^\infty(\Gamma^-)}.$$

Combining (2.22) and (2.24), we can easily deduce the lemma.  $\square$

**2.2.  $L^2$  Estimate.** It is easy to see when  $\lambda \rightarrow 0$ , the estimate in Lemma 2.2 blows up. Hence, we need to show a uniform estimate of the solution to the penalized neutron transport equation (2.15).

**Lemma 2.3.** (*Green's Identity*) Assume  $f(\vec{x}, \vec{w}), g(\vec{x}, \vec{w}) \in L^2(\Omega \times \mathcal{S}^1)$  and  $\vec{w} \cdot \nabla_x f, \vec{w} \cdot \nabla_x g \in L^2(\Omega \times \mathcal{S}^1)$  with  $f, g \in L^2(\Gamma)$ . Then

$$(2.25) \quad \iint_{\Omega \times \mathcal{S}^1} \left( (\vec{w} \cdot \nabla_x f)g + (\vec{w} \cdot \nabla_x g)f \right) d\vec{x} d\vec{w} = \int_{\Gamma} f g d\gamma,$$

where  $d\gamma = (\vec{w} \cdot \vec{n})ds$  on the boundary.

*Proof.* See [5, Chapter 9].  $\square$

**Lemma 2.4.** The solution  $u_\lambda$  to the equation (2.15) satisfies the uniform estimate

$$(2.26) \quad \epsilon \|\bar{u}_\lambda\|_{L^2(\Omega \times \mathcal{S}^1)} \leq C(\Omega) \left( \|u_\lambda - \bar{u}_\lambda\|_{L^2(\Omega \times \mathcal{S}^1)} + \|f\|_{L^2(\Omega \times \mathcal{S}^1)} + \epsilon \|u_\lambda\|_{L^2(\Gamma^+)} + \epsilon \|g\|_{L^2(\Gamma^-)} \right),$$

for  $0 \leq \lambda \ll 1$  and  $0 < \epsilon \ll 1$ .

*Proof.* Applying Lemma 2.3 to the solution of the equation (2.15). Then for any  $\phi \in L^2(\Omega \times \mathcal{S}^1)$  satisfying  $\vec{w} \cdot \nabla_x \phi \in L^2(\Omega \times \mathcal{S}^1)$  and  $\phi \in L^2(\Gamma)$ , we have

$$(2.27) \quad \lambda \iint_{\Omega \times \mathcal{S}^1} u_\lambda \phi + \epsilon \int_{\Gamma} u_\lambda \phi d\gamma - \epsilon \iint_{\Omega \times \mathcal{S}^1} (\vec{w} \cdot \nabla_x \phi) u_\lambda + \iint_{\Omega \times \mathcal{S}^1} (u_\lambda - \bar{u}_\lambda) \phi = \iint_{\Omega \times \mathcal{S}^1} f \phi.$$

Our goal is to choose a particular test function  $\phi$ . We first construct an auxiliary function  $\zeta$ . Since  $u_\lambda \in L^\infty(\Omega \times \mathcal{S}^1)$ , it naturally implies  $\bar{u}_\lambda \in L^\infty(\Omega)$  which further leads to  $\bar{u}_\lambda \in L^2(\Omega)$ . We define  $\zeta(\vec{x})$  on  $\Omega$  satisfying

$$(2.28) \quad \begin{cases} \Delta \zeta &= \bar{u}_\lambda \text{ in } \Omega, \\ \zeta &= 0 \text{ on } \partial\Omega. \end{cases}$$

In the bounded domain  $\Omega$ , based on the standard elliptic estimate, we have

$$(2.29) \quad \|\zeta\|_{H^2(\Omega)} \leq C(\Omega) \|\bar{u}_\lambda\|_{L^2(\Omega)}.$$

We plug the test function

$$(2.30) \quad \phi = -\vec{w} \cdot \nabla_x \zeta$$

into the weak formulation (2.27) and estimate each term there. Naturally, we have

$$(2.31) \quad \|\phi\|_{L^2(\Omega)} \leq C \|\zeta\|_{H^1(\Omega)} \leq C(\Omega) \|\bar{u}_\lambda\|_{L^2(\Omega)}.$$

Easily we can decompose

$$(2.32) \quad -\epsilon \iint_{\Omega \times \mathcal{S}^1} (\vec{w} \cdot \nabla_x \phi) u_\lambda = -\epsilon \iint_{\Omega \times \mathcal{S}^1} (\vec{w} \cdot \nabla_x \phi) \bar{u}_\lambda - \epsilon \iint_{\Omega \times \mathcal{S}^1} (\vec{w} \cdot \nabla_x \phi) (u_\lambda - \bar{u}_\lambda).$$

We estimate the two term on the right-hand side separately. By (2.28) and (2.30), we have

$$\begin{aligned}
 (2.33) \quad -\epsilon \iint_{\Omega \times \mathcal{S}^1} (\vec{w} \cdot \nabla_x \phi) \bar{u}_\lambda &= \epsilon \iint_{\Omega \times \mathcal{S}^1} \bar{u}_\lambda \left( w_1(w_1 \partial_{11} \zeta + w_2 \partial_{12} \zeta) + w_2(w_1 \partial_{12} \zeta + w_2 \partial_{22} \zeta) \right) \\
 &= \epsilon \iint_{\Omega \times \mathcal{S}^1} \bar{u}_\lambda \left( w_1^2 \partial_{11} \zeta + w_2^2 \partial_{22} \zeta \right) \\
 &= \epsilon \pi \int_{\Omega} \bar{u}_\lambda (\partial_{11} \zeta + \partial_{22} \zeta) \\
 &= \epsilon \pi \|\bar{u}_\lambda\|_{L^2(\Omega)}^2 \\
 &= \frac{1}{2} \epsilon \|\bar{u}_\lambda\|_{L^2(\Omega \times \mathcal{S}^1)}^2.
 \end{aligned}$$

In the second equality, above cross terms vanish due to the symmetry of the integral over  $\mathcal{S}^1$ . On the other hand, for the second term in (2.32), Hölder's inequality and the elliptic estimate imply

$$\begin{aligned}
 (2.34) \quad -\epsilon \iint_{\Omega \times \mathcal{S}^1} (\vec{w} \cdot \nabla_x \phi) (u_\lambda - \bar{u}_\lambda) &\leq C(\Omega) \epsilon \|u_\lambda - \bar{u}_\lambda\|_{L^2(\Omega \times \mathcal{S}^1)} \|\zeta\|_{H^2(\Omega)} \\
 &\leq C(\Omega) \epsilon \|u_\lambda - \bar{u}_\lambda\|_{L^2(\Omega \times \mathcal{S}^1)} \|\bar{u}_\lambda\|_{L^2(\Omega \times \mathcal{S}^1)}.
 \end{aligned}$$

Based on (2.29), (2.31), the boundary condition of the penalized neutron transport equation (2.15), the trace theorem, Hölder's inequality and the elliptic estimate, we have

$$\begin{aligned}
 (2.35) \quad \epsilon \int_{\Gamma} u_\lambda \phi d\gamma &= \epsilon \int_{\Gamma^+} u_\lambda \phi d\gamma + \epsilon \int_{\Gamma^-} u_\lambda \phi d\gamma \\
 &\leq C(\Omega) \left( \epsilon \|u_\lambda\|_{L^2(\Gamma^+)} \|\bar{u}_\lambda\|_{L^2(\Omega \times \mathcal{S}^1)} + \epsilon \|g\|_{L^2(\Gamma^-)} \|\bar{u}_\lambda\|_{L^2(\Omega \times \mathcal{S}^1)} \right),
 \end{aligned}$$

$$\begin{aligned}
 (2.36) \quad \lambda \iint_{\Omega \times \mathcal{S}^1} u_\lambda \phi &= \lambda \iint_{\Omega \times \mathcal{S}^1} \bar{u}_\lambda \phi + \lambda \iint_{\Omega \times \mathcal{S}^1} (u_\lambda - \bar{u}_\lambda) \phi = \lambda \iint_{\Omega \times \mathcal{S}^1} (u_\lambda - \bar{u}_\lambda) \phi \\
 &\leq C(\Omega) \lambda \|\bar{u}_\lambda\|_{L^2(\Omega \times \mathcal{S}^1)} \|u_\lambda - \bar{u}_\lambda\|_{L^2(\Omega \times \mathcal{S}^1)},
 \end{aligned}$$

$$(2.37) \quad \iint_{\Omega \times \mathcal{S}^1} (u_\lambda - \bar{u}_\lambda) \phi \leq C(\Omega) \|\bar{u}_\lambda\|_{L^2(\Omega \times \mathcal{S}^1)} \|u_\lambda - \bar{u}_\lambda\|_{L^2(\Omega \times \mathcal{S}^1)},$$

$$(2.38) \quad \iint_{\Omega \times \mathcal{S}^1} f \phi \leq C(\Omega) \|\bar{u}_\lambda\|_{L^2(\Omega \times \mathcal{S}^1)} \|f\|_{L^2(\Omega \times \mathcal{S}^1)}.$$

Collecting terms in (2.33), (2.34), (2.35), (2.36), (2.37) and (2.38), we obtain

$$(2.39) \quad \epsilon \|\bar{u}_\lambda\|_{L^2(\Omega \times \mathcal{S}^1)} \leq C(\Omega) \left( (1 + \epsilon + \lambda) \|u_\lambda - \bar{u}_\lambda\|_{L^2(\Omega \times \mathcal{S}^1)} + \epsilon \|u_\lambda\|_{L^2(\Gamma^+)} + \|f\|_{L^2(\Omega \times \mathcal{S}^1)} + \epsilon \|g\|_{L^2(\Gamma^-)} \right),$$

When  $0 \leq \lambda < 1$  and  $0 < \epsilon < 1$ , we get the desired uniform estimate with respect to  $\lambda$ .  $\square$

**Theorem 2.5.** Assume  $f(\vec{x}, \vec{w}) \in L^\infty(\Omega \times \mathcal{S}^1)$  and  $g(x_0, \vec{w}) \in L^\infty(\Gamma^-)$ . Then for the steady neutron transport equation (2.1), there exists a unique solution  $u(\vec{x}, \vec{w}) \in L^2(\Omega \times \mathcal{S}^1)$  satisfying

$$(2.40) \quad \|u\|_{L^2(\Omega \times \mathcal{S}^1)} \leq C(\Omega) \left( \frac{1}{\epsilon^2} \|f\|_{L^2(\Omega \times \mathcal{S}^1)} + \frac{1}{\epsilon^{1/2}} \|g\|_{L^2(\Gamma^-)} \right).$$

*Proof.* In the weak formulation (2.27), we may take the test function  $\phi = u_\lambda$  to get the energy estimate

$$(2.41) \quad \lambda \|u_\lambda\|_{L^2(\Omega \times \mathcal{S}^1)}^2 + \frac{1}{2} \epsilon \int_{\Gamma} |u_\lambda|^2 d\gamma + \|u_\lambda - \bar{u}_\lambda\|_{L^2(\Omega \times \mathcal{S}^1)}^2 = \iint_{\Omega \times \mathcal{S}^1} f u_\lambda.$$

Hence, this naturally implies

$$(2.42) \quad \frac{1}{2} \epsilon \|u_\lambda\|_{L^2(\Gamma^+)}^2 + \|u_\lambda - \bar{u}_\lambda\|_{L^2(\Omega \times \mathcal{S}^1)}^2 \leq \iint_{\Omega \times \mathcal{S}^1} f u_\lambda + \frac{1}{2} \epsilon \|g\|_{L^2(\Gamma^-)}^2.$$

On the other hand, we can square on both sides of (2.26) to obtain

$$(2.43) \quad \epsilon^2 \|\bar{u}_\lambda\|_{L^2(\Omega \times \mathcal{S}^1)}^2 \leq C(\Omega) \left( \|u_\lambda - \bar{u}_\lambda\|_{L^2(\Omega \times \mathcal{S}^1)}^2 + \|f\|_{L^2(\Omega \times \mathcal{S}^1)}^2 + \epsilon^2 \|u_\lambda\|_{L^2(\Gamma^+)}^2 + \epsilon^2 \|g\|_{L^2(\Gamma^-)}^2 \right).$$

Multiplying a sufficiently small constant on both sides of (2.43) and adding it to (2.42) to absorb  $\|u_\lambda\|_{L^2(\Gamma^+)}^2$  and  $\|u_\lambda - \bar{u}_\lambda\|_{L^2(\Omega \times \mathcal{S}^1)}^2$ , we deduce

$$(2.44) \quad \begin{aligned} & \epsilon \|u_\lambda\|_{L^2(\Gamma^+)}^2 + \epsilon^2 \|\bar{u}_\lambda\|_{L^2(\Omega \times \mathcal{S}^1)}^2 + \|u_\lambda - \bar{u}_\lambda\|_{L^2(\Omega \times \mathcal{S}^1)}^2 \\ & \leq C(\Omega) \left( \|f\|_{L^2(\Omega \times \mathcal{S}^1)}^2 + \iint_{\Omega \times \mathcal{S}^1} f u_\lambda + \epsilon \|g\|_{L^2(\Gamma^-)}^2 \right). \end{aligned}$$

Hence, we have

$$(2.45) \quad \epsilon \|u_\lambda\|_{L^2(\Gamma^+)}^2 + \epsilon^2 \|u_\lambda\|_{L^2(\Omega \times \mathcal{S}^1)}^2 \leq C(\Omega) \left( \|f\|_{L^2(\Omega \times \mathcal{S}^1)}^2 + \iint_{\Omega \times \mathcal{S}^1} f u_\lambda + \epsilon \|g\|_{L^2(\Gamma^-)}^2 \right).$$

A simple application of Cauchy's inequality leads to

$$(2.46) \quad \iint_{\Omega \times \mathcal{S}^1} f u_\lambda \leq \frac{1}{4C\epsilon^2} \|f\|_{L^2(\Omega \times \mathcal{S}^1)}^2 + C\epsilon^2 \|u_\lambda\|_{L^2(\Omega \times \mathcal{S}^1)}^2.$$

Taking  $C$  sufficiently small, we can divide (2.45) by  $\epsilon^2$  to obtain

$$(2.47) \quad \frac{1}{\epsilon} \|u_\lambda\|_{L^2(\Gamma^+)}^2 + \|u_\lambda\|_{L^2(\Omega \times \mathcal{S}^2)}^2 \leq C(\Omega) \left( \frac{1}{\epsilon^4} \|f\|_{L^2(\Omega \times \mathcal{S}^2)}^2 + \frac{1}{\epsilon} \|g\|_{L^2(\Gamma^-)}^2 \right).$$

Since above estimate does not depend on  $\lambda$ , it gives a uniform estimate for the penalized neutron transport equation (2.15). Thus, we can extract a weakly convergent subsequence  $u_\lambda \rightarrow u$  as  $\lambda \rightarrow 0$ . The weak lower semi-continuity of norms  $\|\cdot\|_{L^2(\Omega \times \mathcal{S}^2)}$  and  $\|\cdot\|_{L^2(\Gamma^+)}$  implies  $u$  also satisfies the estimate (2.47). Hence, in the weak formulation (2.27), we can take  $\lambda \rightarrow 0$  to deduce that  $u$  satisfies equation (2.1). Also  $u_\lambda - u$  satisfies the equation

$$(2.48) \quad \begin{cases} \epsilon \vec{w} \cdot \nabla_x (u_\lambda - u) + (u_\lambda - u) - (\bar{u}_\lambda - \bar{u}) &= -\lambda u_\lambda \text{ in } \Omega, \\ (u_\lambda - u)(\vec{x}_0, \vec{w}) &= 0 \text{ for } \vec{x}_0 \in \partial\Omega \text{ and } \vec{w} \cdot \vec{n} < 0. \end{cases}$$

By a similar argument as above, we can achieve

$$(2.49) \quad \|u_\lambda - u\|_{L^2(\Omega \times \mathcal{S}^2)}^2 \leq C(\Omega) \left( \frac{\lambda}{\epsilon^4} \|u_\lambda\|_{L^2(\Omega \times \mathcal{S}^2)}^2 \right).$$

When  $\lambda \rightarrow 0$ , the right-hand side approaches zero, which implies the convergence is actually in the strong sense. The uniqueness easily follows from the energy estimates.  $\square$

### 2.3. $L^\infty$ Estimate.

**Theorem 2.6.** *Assume  $f(\vec{x}, \vec{w}) \in L^\infty(\Omega \times \mathcal{S}^1)$  and  $g(x_0, \vec{w}) \in L^\infty(\Gamma^-)$ . Then for the neutron transport equation (2.1), there exists a unique solution  $u(\vec{x}, \vec{w}) \in L^\infty(\Omega \times \mathcal{S}^1)$  satisfying*

$$(2.50) \quad \|u\|_{L^\infty(\Omega \times \mathcal{S}^1)} \leq C(\Omega) \left( \frac{1}{\epsilon^3} \|f\|_{L^\infty(\Omega \times \mathcal{S}^1)} + \frac{1}{\epsilon^{3/2}} \|g\|_{L^\infty(\Gamma^-)} \right).$$

*Proof.* We divide the proof into several steps to bootstrap an  $L^2$  solution to an  $L^\infty$  solution:

Step 1: Double Duhamel iterations.

The characteristics of the equation (2.1) is given by (2.10). Hence, we can rewrite the equation (2.1) along the characteristics as

$$(2.51) \quad \begin{aligned} u(\vec{x}, \vec{w}) &= g(\vec{x} - \epsilon t_b \vec{w}, \vec{w}) e^{-t_b} + \int_0^{t_b} f(\vec{x} - \epsilon(t_b - s) \vec{w}, \vec{w}) e^{-(t_b - s)} ds \\ &\quad + \frac{1}{2\pi} \int_0^{t_b} \left( \int_{\mathcal{S}^1} u(\vec{x} - \epsilon(t_b - s) \vec{w}, \vec{w}_t) d\vec{w}_t \right) e^{-(t_b - s)} ds. \end{aligned}$$

where the backward exit time  $t_b$  is defined as (2.13). Note we have replaced  $\bar{u}$  by the integral of  $u$  over the dummy velocity variable  $\vec{w}_t$ . For the last term in this formulation, we apply the Duhamel's principle again to  $u(\vec{x} - \epsilon(t_b - s)\vec{w}, \vec{w}_t)$  and obtain

(2.52)

$$\begin{aligned} u(\vec{x}, \vec{w}) &= g(\vec{x} - \epsilon t_b \vec{w}, \vec{w}) e^{-t_b} + \int_0^{t_b} f(\vec{x} - \epsilon(t_b - s)\vec{w}, \vec{w}) e^{-(t_b - s)} ds \\ &\quad + \frac{1}{2\pi} \int_0^{t_b} \int_{S^1} g(\vec{x} - \epsilon(t_b - s)\vec{w} - \epsilon s_b \vec{w}_t, \vec{w}_t) e^{-s_b} d\vec{w}_t e^{-(t_b - s)} ds \\ &\quad + \frac{1}{2\pi} \int_0^{t_b} \int_{S^1} \left( \int_0^{s_b} f(\vec{x} - \epsilon(t_b - s)\vec{w} - \epsilon(s_b - r)\vec{w}_t, \vec{w}_t) e^{-(s_b - r)} dr \right) d\vec{w}_t e^{-(t_b - s)} ds \\ &\quad + \left( \frac{1}{2\pi} \right)^2 \int_0^{t_b} \int_{S^1} e^{-(t_b - s)} \left( \int_0^{s_b} \int_{S^1} e^{-(s_b - r)} u(\vec{x} - \epsilon(t_b - s)\vec{w} - \epsilon(s_b - r)\vec{w}_t, \vec{w}_s) d\vec{w}_s dr \right) d\vec{w}_t ds, \end{aligned}$$

where we introduce another dummy velocity variable  $\vec{w}_s$  and

$$(2.53) \quad s_b(\vec{x}, \vec{w}, s, \vec{w}_t) = \inf\{r \geq 0 : (\vec{x} - \epsilon(t_b - s)\vec{w} - \epsilon r \vec{w}_t, \vec{w}_t) \in \Gamma^-\}.$$

Step 2: Estimates of all but the last term in (2.52).

We can directly estimate as follows:

$$(2.54) \quad |g(\vec{x} - \epsilon t_b \vec{w}, \vec{w}) e^{-t_b}| \leq \|g\|_{L^\infty(\Gamma^-)},$$

$$(2.55) \quad \left| \frac{1}{2\pi} \int_0^{t_b} \int_{S^1} g(\vec{x} - \epsilon(t_b - s)\vec{w} - \epsilon s_b \vec{w}_t, \vec{w}_t) e^{-s_b} d\vec{w}_t e^{-(t_b - s)} ds \right| \leq \|g\|_{L^\infty(\Gamma^-)},$$

$$(2.56) \quad \left| \int_0^{t_b} f(\vec{x} - \epsilon(t_b - s)\vec{w}, \vec{w}) e^{-(t_b - s)} ds \right| \leq \|f\|_{L^\infty(\Omega \times S^1)},$$

(2.57)

$$\left| \frac{1}{2\pi} \int_0^{t_b} \int_{S^1} \left( \int_0^{s_b} f(\vec{x} - \epsilon(t_b - s)\vec{w} - \epsilon(s_b - r)\vec{w}_t, \vec{w}_t) e^{-(s_b - r)} dr \right) d\vec{w}_t e^{-(t_b - s)} ds \right| \leq \|f\|_{L^\infty(\Omega \times S^1)}.$$

Step 3: Estimates of the last term in (2.52).

Now we decompose the last term in (2.52) as

$$(2.58) \quad \int_0^{t_b} \int_{S^1} \int_0^{s_b} \int_{S^1} = \int_0^{t_b} \int_{S^1} \int_{s_b - r \leq \delta} \int_{S^1} + \int_0^{t_b} \int_{S^1} \int_{s_b - r \geq \delta} \int_{S^1} = I_1 + I_2,$$

for some  $\delta > 0$ . We can estimate  $I_1$  directly as

$$(2.59) \quad I_1 \leq \int_0^{t_b} e^{-(t_b - s)} \left( \int_{\max(0, s_b - \delta)}^{s_b} \|u\|_{L^\infty(\Omega \times S^1)} dr \right) ds \leq \delta \|u\|_{L^\infty(\Omega \times S^1)}.$$

Then we can bound  $I_2$  as

$$(2.60) \quad I_2 \leq C \int_0^{t_b} \int_{S^1} \int_0^{\max(0, s_b - \delta)} \int_{S^1} |u(\vec{x} - \epsilon(t_b - s)\vec{w} - \epsilon(s_b - r)\vec{w}_t, \vec{w}_s)| e^{-(t_b - s)} d\vec{w}_s dr d\vec{w}_t ds.$$

By the definition of  $t_b$  and  $s_b$ , we always have  $\vec{x} - \epsilon(t_b - s)\vec{w} - \epsilon(s_b - r)\vec{w}_t \in \bar{\Omega}$ . Hence, we may interchange the order of integration and apply Hölder's inequality to obtain

$$\begin{aligned}
 (2.61) \quad I_2 &\leq C \int_0^{t_b} \int_0^{\max(0, s_b - \delta)} \int_{S^1} \int_{S^1} \mathbf{1}_{\Omega}(\vec{x} - \epsilon(t_b - s)\vec{w} - \epsilon(s_b - r)\vec{w}_t) \\
 &\quad |u(\vec{x} - \epsilon(t_b - s)\vec{w} - \epsilon(s_b - r)\vec{w}_t, \vec{w}_s)| e^{-(t_b - s)} d\vec{w}_t d\vec{w}_s dr ds \\
 &\leq C \int_0^{t_b} \int_{S^1} \left( \int_0^{\max(0, s_b - \delta)} \int_{S^1} \mathbf{1}_{\Omega}(\vec{x} - \epsilon(t_b - s)\vec{w} - \epsilon(s_b - r)\vec{w}_t) \right. \\
 &\quad \left. |u(\vec{x} - \epsilon(t_b - s)\vec{w} - \epsilon(s_b - r)\vec{w}_t, \vec{w}_s)|^2 d\vec{w}_t dr \right)^{1/2} e^{-(t_b - s)} d\vec{w}_s ds.
 \end{aligned}$$

Note  $\vec{w}_t \in S^1$ , which is essentially a one-dimensional variable. Thus, we may write it in a new variable  $\psi$  as  $\vec{w}_t = (\cos \psi, \sin \psi)$ . Then we define the change of variable  $[-\pi, \pi) \times \mathbb{R} \rightarrow \Omega : (\psi, r) \rightarrow (y_1, y_2) = \vec{y} = \vec{x} - \epsilon(t_b - s)\vec{w} - \epsilon(s_b - r)\vec{w}_t$ , i.e.

$$\begin{aligned}
 (2.62) \quad \begin{cases} y_1 &= x_1 - \epsilon(t_b - s)w_1 - \epsilon(s_b - r) \cos \psi, \\ y_2 &= x_2 - \epsilon(t_b - s)w_2 - \epsilon(s_b - r) \sin \psi. \end{cases}
 \end{aligned}$$

Therefore, for  $s_b - r \geq \delta$ , we can directly compute the Jacobian

$$(2.63) \quad \left| \frac{\partial(y_1, y_2)}{\partial(\psi, r)} \right| = \left\| \begin{pmatrix} -\epsilon(s_b - r) \sin \psi & \epsilon \cos \psi \\ \epsilon(s_b - r) \cos \psi & \epsilon \sin \psi \end{pmatrix} \right\| = \epsilon^2(s_b - r) \geq \epsilon^2 \delta.$$

Hence, we may simplify (2.61) as

$$(2.64) \quad I_2 \leq \frac{C}{\epsilon \sqrt{\delta}} \int_0^{t_b} \int_{S^1} \left( \int_{\Omega} |u(\vec{y}, \vec{w}_s)|^2 d\vec{y} \right)^{1/2} e^{-(t_b - s)} d\vec{w}_s ds.$$

Then we may further utilize Cauchy's inequality and the  $L^2$  estimate of  $u$  in Theorem 2.5 to obtain

$$\begin{aligned}
 (2.65) \quad I_2 &\leq \frac{C}{\epsilon \sqrt{\delta}} \int_0^{t_b} \left( \int_{S^1} \int_{\Omega} |u(\vec{y}, \vec{w}_s)|^2 d\vec{y} d\vec{w}_s \right)^{1/2} e^{-(t_b - s)} ds \\
 &= \frac{C}{\epsilon \sqrt{\delta}} \int_0^{t_b} e^{-(t_b - s)} \|u\|_{L^2(\Omega \times S^1)} ds \\
 &\leq \frac{C}{\epsilon \sqrt{\delta}} \|u\|_{L^2(\Omega \times S^1)} \\
 &\leq \frac{C(\Omega)}{\sqrt{\delta}} \left( \frac{1}{\epsilon^3} \|f\|_{L^2(\Omega \times S^1)} + \frac{1}{\epsilon^{3/2}} \|g\|_{L^2(\Gamma^-)} \right) \\
 &\leq \frac{C(\Omega)}{\sqrt{\delta}} \left( \frac{1}{\epsilon^3} \|f\|_{L^\infty(\Omega \times S^1)} + \frac{1}{\epsilon^{3/2}} \|g\|_{L^\infty(\Gamma^-)} \right).
 \end{aligned}$$

In summary, collecting (2.54), (2.55), (2.56), (2.57), (2.59) and (2.65), for fixed  $0 < \delta < 1$ , we have

$$(2.66) \quad |u(\vec{x}, \vec{w})| \leq \delta \|u\|_{L^\infty(\Omega \times S^1)} + \frac{C(\Omega)}{\sqrt{\delta}} \left( \frac{1}{\epsilon^3} \|f\|_{L^\infty(\Omega \times S^1)} + \frac{1}{\epsilon^{3/2}} \|g\|_{L^\infty(\Gamma^-)} \right).$$

Then we may take  $0 < \delta \leq 1/2$  to obtain

$$(2.67) \quad |u(\vec{x}, \vec{w})| \leq \frac{1}{2} \|u\|_{L^\infty(\Omega \times S^1)} + \frac{C(\Omega)}{\sqrt{\delta}} \left( \frac{1}{\epsilon^3} \|f\|_{L^\infty(\Omega \times S^1)} + \frac{1}{\epsilon^{3/2}} \|g\|_{L^\infty(\Gamma^-)} \right).$$

Taking supremum of  $u$  over all  $(\vec{x}, \vec{w})$ , we have

$$(2.68) \quad \|u\|_{L^\infty(\Omega \times S^1)} \leq \frac{1}{2} \|u\|_{L^\infty(\Omega \times S^1)} + \frac{C(\Omega)}{\sqrt{\delta}} \left( \frac{1}{\epsilon^3} \|f\|_{L^\infty(\Omega \times S^1)} + \frac{1}{\epsilon^{3/2}} \|g\|_{L^\infty(\Gamma^-)} \right).$$

Finally, absorbing  $\|u\|_{L^\infty(\Omega \times S^1)}$ , for fixed  $0 < \delta \leq 1/2$ , we get

$$(2.69) \quad \|u\|_{L^\infty(\Omega \times S^1)} \leq C(\Omega) \left( \frac{1}{\epsilon^3} \|f\|_{L^\infty(\Omega \times S^1)} + \frac{1}{\epsilon^{3/2}} \|g\|_{L^\infty(\Gamma^-)} \right).$$

□

#### 2.4. Well-posedness of Transport Equation.

**Theorem 2.7.** *Assume  $g(x_0, \vec{w}) \in L^\infty(\Gamma^-)$ . Then for the steady neutron transport equation (1), there exists a unique solution  $u^\epsilon(\vec{x}, \vec{w}) \in L^\infty(\Omega \times \mathcal{S}^1)$  satisfying*

$$(2.70) \quad \|u^\epsilon\|_{L^\infty(\Omega \times \mathcal{S}^1)} \leq C(\Omega) \frac{1}{\epsilon^{3/2}} \|g\|_{L^\infty(\Gamma^-)}.$$

*Proof.* We can apply Theorem 2.6 to the equation (1). The result naturally follows.  $\square$

#### 3. $\epsilon$ -MILNE PROBLEM

We consider the  $\epsilon$ -Milne problem for  $f^\epsilon(\eta, \theta, \phi)$  in the domain  $(\eta, \theta, \phi) \in [0, \infty) \times [-\pi, \pi) \times [-\pi, \pi)$

$$(3.1) \quad \begin{cases} \sin \phi \frac{\partial f^\epsilon}{\partial \eta} + F(\epsilon; \eta) \cos \phi \frac{\partial f^\epsilon}{\partial \phi} + f^\epsilon - \bar{f}^\epsilon &= S^\epsilon(\eta, \theta, \phi), \\ f^\epsilon(0, \theta, \phi) &= h^\epsilon(\theta, \phi) \text{ for } \sin \phi > 0, \\ \lim_{\eta \rightarrow \infty} f^\epsilon(\eta, \theta, \phi) &= f_\infty^\epsilon(\theta), \end{cases}$$

where

$$(3.2) \quad \bar{f}^\epsilon(\eta, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f^\epsilon(\eta, \theta, \phi) d\phi,$$

$F(\epsilon; \eta)$  is defined as (1.48),

$$(3.3) \quad |h^\epsilon(\theta, \phi)| \leq M,$$

and

$$(3.4) \quad |S^\epsilon(\eta, \theta, \phi)| \leq C e^{-K\eta},$$

for  $M$  and  $K$  uniform in  $\epsilon$  and  $\theta$ .

We may further define a potential function  $V(\epsilon; \eta)$  satisfying  $V(\epsilon; 0) = 0$  and  $F(\epsilon; \eta) = -\partial_\eta V(\epsilon; \eta)$ . The following lemma illustrates the main properties of  $F$  and  $V$ .

**Lemma 3.1.**  *$V(\epsilon; \eta)$  is monotonically increasing with respect to  $\eta$  and satisfies*

$$(3.5) \quad 0 \leq V(\epsilon; \eta) \leq \ln 4.$$

*Define*

$$(3.6) \quad V_\infty(\epsilon) = \lim_{\eta \rightarrow \infty} V(\epsilon; \eta).$$

*Then  $V_\infty(\epsilon) = V_\infty$  independent of  $\epsilon$ . For any  $0 \leq y \leq z \leq \infty$*

$$(3.7) \quad -\ln 4 \leq \int_y^z F(\epsilon; \eta) d\eta \leq 0.$$

*For any  $\sigma > 0$  and  $\eta \geq 0$ , we have*

$$(3.8) \quad e^{V(\eta+\sigma)-V(\eta)} \leq 1 + 4\epsilon\sigma.$$

*Moreover, we have*

$$(3.9) \quad \int_0^\infty \int_\eta^\infty |F(y)|^2 dy d\eta \leq 3 - \ln 4.$$

$$(3.10) \quad \int_0^\infty |F(\epsilon; \eta)|^2 d\eta \leq 3\epsilon.$$

$$(3.11) \quad \|F(\epsilon; \eta)\|_{L^\infty} \leq 4\epsilon.$$

*Proof.* Since  $F(\epsilon; \eta) \leq 0$ , by definition, we know  $V(\epsilon; \eta)$  is monotonically increasing with respect to  $\eta$ . Since  $V(\epsilon; 0) = 0$ ,  $V(\epsilon; \eta) \geq 0$ . Then based on (1.26), we can direct estimate

$$(3.12) \quad \begin{aligned} V(\epsilon; \eta) &= V(\epsilon; 0) - \int_0^\eta F(\epsilon; y) dy = \int_0^\eta \frac{\epsilon \psi(\epsilon y)}{1 - \epsilon y} dy \\ &\leq \int_0^{\frac{3}{4\epsilon}} \frac{\epsilon}{1 - \epsilon y} dy = -\ln(1 - \epsilon y) \Big|_{y=0}^{y=\frac{3}{4\epsilon}} = \ln 4. \end{aligned}$$

This verifies (3.5).

Also, letting  $\mu = \epsilon \eta$ , we have

$$(3.13) \quad V_\infty(\epsilon) = - \int_0^\infty \frac{\epsilon \psi(\epsilon \eta)}{1 - \epsilon \eta} d\eta = - \int_0^\infty \frac{\psi(\epsilon \eta)}{1 - \epsilon \eta} d(\epsilon \eta) = - \int_0^\infty \frac{\psi(\mu)}{1 - \mu} d\mu \leq \ln 4.$$

Hence, we have  $V_\infty(\epsilon) = V_\infty$  independent of  $\epsilon$ . Since

$$(3.14) \quad \int_y^z F(\epsilon; \eta) d\eta = V(\epsilon; y) - V(\epsilon; z),$$

we can naturally obtain (3.7).

Moreover, we have

$$(3.15) \quad V(\eta + \sigma) - V(\eta) = - \int_\eta^{\eta + \sigma} F(\epsilon; y) dy = \int_\eta^{\eta + \sigma} \frac{\epsilon \psi(\epsilon y)}{1 - \epsilon y} dy$$

$$(3.16) \quad \leq \int_\eta^{\min\{\eta + \sigma, \frac{3}{4\epsilon}\}} \frac{\epsilon \psi(\epsilon y)}{1 - \epsilon y} dy \leq -\ln(1 - \epsilon y) \Big|_{y=\eta}^{y=\min\{\eta + \sigma, \frac{3}{4\epsilon}\}}.$$

If  $\eta + \sigma \leq \frac{3}{4\epsilon}$ , we have

$$(3.17) \quad e^{V(\eta + \sigma) - V(\eta)} \leq \frac{1 - \epsilon \eta}{1 - \epsilon(\eta + \sigma)} = 1 + \frac{\epsilon \sigma}{1 - \epsilon(\eta + \sigma)} \leq 1 + 4\epsilon \sigma.$$

On the other hand, if  $\eta + \sigma \geq \frac{3}{4\epsilon}$  and  $\eta \leq \frac{3}{4\epsilon}$ , we define  $\sigma' = \frac{3}{4\epsilon} - \eta$  to obtain

$$(3.18) \quad e^{V(\eta + \sigma) - V(\eta)} \leq e^{V(\eta + \sigma') - V(\eta)} \leq 1 + 4\epsilon \sigma' \leq 1 + 4\epsilon \sigma.$$

Finally, if  $\eta + \sigma \geq \frac{3}{4\epsilon}$  and  $\eta \geq \frac{3}{4\epsilon}$ , we have

$$(3.19) \quad e^{V(\eta + \sigma) - V(\eta)} = e^0 = 1.$$

Therefore, (3.8) follows.

Furthermore, considering the cut-off function (1.26), we have

$$(3.20) \quad \begin{aligned} \int_0^\infty \int_\eta^\infty |F(y)|^2 dy d\eta &\leq \int_0^\infty \int_\eta^\infty \left| \frac{\epsilon \psi(\epsilon y)}{1 - \epsilon y} \right|^2 dy d\eta \leq \int_0^{\frac{3}{4\epsilon}} \int_\eta^{\frac{3}{4\epsilon}} \frac{\epsilon^2}{(1 - \epsilon y)^2} dy d\eta \\ &= \int_0^{\frac{3}{4\epsilon}} \left( \frac{\epsilon}{1 - \epsilon y} \Big|_{y=\eta}^{y=\frac{3}{4\epsilon}} \right) d\eta = \int_0^{\frac{3}{4\epsilon}} \left( 4\epsilon - \frac{\epsilon}{1 - \epsilon \eta} \right) d\eta \\ &= \left( 4\epsilon \eta + \ln(1 - \epsilon \eta) \right) \Big|_{\eta=0}^{\eta=\frac{3}{4\epsilon}} = 3 - \ln 4. \end{aligned}$$

Also, we can directly estimate

$$(3.21) \quad \begin{aligned} \int_0^\infty |F(\epsilon; \eta)|^2 d\eta &= \int_0^\infty \frac{\epsilon^2 \psi^2(\epsilon \eta)}{(1 - \epsilon \eta)^2} d\eta \leq \int_0^{\frac{3}{4\epsilon}} \frac{\epsilon^2}{(1 - \epsilon \eta)^2} d\eta \\ &= \frac{\epsilon}{1 - \epsilon \eta} \Big|_{\eta=0}^{\eta=\frac{3}{4\epsilon}} = 3\epsilon. \end{aligned}$$

Finally, based on the definition of  $F$  and  $\psi$ , we obtain

$$(3.22) \quad |F(\epsilon; \eta)| = \left| \frac{\epsilon \psi(\epsilon \eta)}{1 - \epsilon \eta} \right| \leq \frac{\epsilon}{1 - \epsilon \frac{3}{4\epsilon}} = 4\epsilon.$$



□

For notational simplicity, we omit  $\epsilon$  and  $\theta$  dependence in  $f^\epsilon$  in this section. The same convention also applies to  $F(\epsilon; \eta)$ ,  $V(\epsilon; \eta)$ ,  $S^\epsilon(\eta, \theta, \phi)$  and  $h^\epsilon(\theta, \phi)$ . However, our estimates are independent of  $\epsilon$  and  $\theta$ .

In this section, we introduce some special notations to describe the norms in the space  $(\eta, \phi) \in [0, \infty) \times [-\pi, \pi)$ . Define the  $L^2$  norm as follows:

$$(3.23) \quad \|f(\eta)\|_{L^2} = \left( \int_{-\pi}^{\pi} |f(\eta, \phi)|^2 d\phi \right)^{1/2},$$

$$(3.24) \quad \|f\|_{L^2 L^2} = \left( \int_0^\infty \int_{-\pi}^{\pi} |f(\eta, \phi)|^2 d\phi d\eta \right)^{1/2}.$$

Define the inner product in  $\phi$  space

$$(3.25) \quad \langle f, g \rangle_\phi(\eta) = \int_{-\pi}^{\pi} f(\eta, \phi) g(\eta, \phi) d\phi.$$

Define the  $L^\infty$  norm as follows:

$$(3.26) \quad \|f(\eta)\|_{L^\infty} = \sup_{\phi \in [-\pi, \pi)} |f(\eta, \phi)|,$$

$$(3.27) \quad \|f\|_{L^\infty L^\infty} = \sup_{(\eta, \phi) \in [0, \infty) \times [-\pi, \pi)} |f(\eta, \phi)|,$$

$$(3.28) \quad \|f\|_{L^\infty L^2} = \sup_{\eta \in [0, \infty)} \left( \int_{-\pi}^{\pi} |f(\eta, \phi)|^2 d\phi \right)^{1/2}.$$

Since the boundary data  $h(\phi)$  is only defined on  $\sin \phi > 0$ , we naturally extend above definitions on this half-domain as follows:

$$(3.29) \quad \|h\|_{L^2} = \left( \int_{\sin \phi > 0} |h(\phi)|^2 d\phi \right)^{1/2},$$

$$(3.30) \quad \|h\|_{L^\infty} = \sup_{\sin \phi > 0} |h(\phi)|.$$

**Lemma 3.2.** *We have*

$$(3.31) \quad \|h\|_{L^2} \leq C \|h\|_{L^\infty} \leq CM$$

$$(3.32) \quad \|S\|_{L^2 L^2} \leq C \frac{M}{K}$$

$$(3.33) \quad \|S\|_{L^\infty L^2} \leq C \|S\|_{L^\infty L^\infty} \leq CM$$

*Proof.* They can be verified via direct computation, so we omit the proofs here. □

### 3.1. $L^2$ Estimates.

3.1.1. *Finite Slab with  $\bar{S} = 0$ .* Consider the  $\epsilon$ -Milne problem for  $f^L(\eta, \phi)$  in a finite slab  $(\eta, \phi) \in [0, L] \times [-\pi, \pi)$

$$(3.34) \quad \begin{cases} \sin \phi \frac{\partial f^L}{\partial \eta} + F(\eta) \cos \phi \frac{\partial f^L}{\partial \phi} + f^L - \bar{f}^L = S(\eta, \phi), \\ f^L(0, \phi) = h(\phi) \text{ for } \sin \phi < 0, \\ f^L(L, \phi) = f^L(L, R\phi), \end{cases}$$

where  $R\phi = -\phi$  and  $S$  satisfies  $\bar{S}(\eta) = 0$  for any  $\eta$ . We may decompose the solution

$$(3.35) \quad f^L(\eta, \phi) = q_f^L(\eta) + r_f^L(\eta, \phi),$$

where the hydrodynamical part  $q_f^L$  is in the null space of the operator  $f^L - \bar{f}^L$ , and the microscopic part  $r_f^L$  is the orthogonal complement, i.e.

$$(3.36) \quad q_f^L(\eta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f^L(\eta, \phi) d\phi \quad r_f^L(\eta, \phi) = f^L(\eta, \phi) - q_f^L(\eta).$$

In the following, when there is no confusion, we simply write  $f^L = q^L + r^L$ .

**Lemma 3.3.** *Assume  $\bar{S}(\eta) = 0$  for any  $\eta \in [0, L]$  with (3.3) and (3.4). Then there exists a solution  $f(\eta, \phi)$  to the finite slab problem (3.34) satisfying*

$$(3.37) \quad \int_0^L \|r^L(\eta)\|_{L^2}^2 d\eta \leq C \left( M + \frac{M}{K} \right)^2 < \infty,$$

$$(3.38) \quad \|q^L(\eta)\|_{L^2}^2 \leq C \left( 1 + M + \frac{M}{K} \right)^2 \left( 1 + \eta^{1/2} + \|r^L(\eta)\|_{L^2} \right),$$

$$(3.39) \quad \langle \sin \phi, r^L \rangle_\phi(\eta) = 0,$$

for arbitrary  $\eta \in [0, L]$ .

*Proof.* We divide the proof into several steps:

Step 1: Assume  $\|H\|_{L^\infty L^\infty} < \infty$  and  $\|h\|_{L^\infty} < \infty$ , then the solution  $f_\lambda(\eta, \phi)$  to the penalized  $\epsilon$ -transport equation

$$(3.40) \quad \begin{cases} \lambda f_\lambda + \sin \phi \frac{\partial f_\lambda}{\partial \eta} + F(\eta) \cos \phi \frac{\partial f_\lambda}{\partial \phi} + f_\lambda = H(\eta, \phi), \\ f_\lambda(0, \phi) = h(\phi) \text{ for } \sin \phi < 0, \\ f_\lambda(L, \phi) = f_\lambda(L, R\phi). \end{cases}$$

satisfies

$$(3.41) \quad \|f_\lambda\|_{L^\infty L^\infty} \leq \|h\|_{L^\infty} + \|H\|_{L^\infty L^\infty}.$$

*The proof of (3.41):* To construct the solution of (3.40), we define the energy as

$$(3.42) \quad E(\eta, \phi) = \cos \phi e^{-V(\eta)}.$$

This curve with constant energy is the characteristics of the equation (3.40). Hence, on this curve the equation can be simplified as follows:

$$(3.43) \quad \lambda f_\lambda + \sin \phi \frac{df_\lambda}{d\eta} + f_\lambda = H.$$

An implicit function  $\eta^+(\eta, \phi)$  can be determined through

$$(3.44) \quad |E(\eta, \phi)| = e^{-V(\eta^+)}.$$

which means  $(\eta^+, \phi_0)$  with  $\sin \phi_0 = 0$  is on the same characteristics as  $(\eta, \phi)$ . Define the quantities for  $0 \leq \eta' \leq \eta^+$  as follows:

$$(3.45) \quad \phi'(\phi, \eta, \eta') = \cos^{-1}(\cos \phi e^{V(\eta') - V(\eta)}),$$

$$(3.46) \quad R\phi'(\phi, \eta, \eta') = -\cos^{-1}(\cos \phi e^{V(\eta') - V(\eta)}) = -\phi'(\phi, \eta, \eta'),$$

where the inverse trigonometric function can be defined single-valued in the domain  $[0, \pi)$  and the quantities are always well-defined due to the monotonicity of  $V$ . Finally we put

$$(3.47) \quad G_{\eta, \eta'}^\lambda(\phi) = \int_{\eta'}^\eta \frac{1 + \lambda}{\sin(\phi'(\phi, \eta, \xi))} d\xi.$$

We can define the solution to (3.40) along the characteristics as follows:

Case I:

For  $\sin \phi > 0$ ,

$$(3.48) \quad f_\lambda(\eta, \phi) = h(\phi'(\phi, \eta, 0)) \exp(-G_{\eta, 0}^\lambda) + \int_0^\eta \frac{H(\eta', \phi'(\phi, \eta, \eta'))}{\sin(\phi'(\phi, \eta, \eta'))} \exp(-G_{\eta, \eta'}^\lambda) d\eta'.$$

Case II:

For  $\sin \phi < 0$  and  $|E(\eta, \phi)| \leq e^{-V(L)}$ ,

$$(3.49) \quad \begin{aligned} f_\lambda(\eta, \phi) &= h(\phi'(\phi, \eta, 0)) \exp(-G_{L, 0}^\lambda - G_{L, \eta}^\lambda) \\ &+ \left( \int_0^L \frac{H(\eta', \phi'(\phi, \eta, \eta'))}{\sin(\phi'(\phi, \eta, \eta'))} \exp(-G_{L, \eta'}^\lambda - G_{L, \eta}^\lambda) d\eta' + \int_\eta^L \frac{H(\eta', R\phi'(\phi, \eta, \eta'))}{\sin(\phi'(\phi, \eta, \eta'))} \exp(G_{\eta, \eta'}^\lambda) d\eta' \right). \end{aligned}$$

Case III:

For  $\sin \phi < 0$  and  $|E(\eta, \phi)| \geq e^{-V(L)}$ ,

(3.50)

$$\begin{aligned} f_\lambda(\eta, \phi) &= h(\phi'(\phi, \eta, 0)) \exp(-G_{\eta^+, 0}^\lambda - G_{\eta^+, \eta}^\lambda) \\ &+ \left( \int_0^{\eta^+} \frac{H(\eta', \phi'(\phi, \eta, \eta'))}{\sin(\phi'(\phi, \eta, \eta'))} \exp(-G_{\eta^+, \eta'}^\lambda - G_{\eta^+, \eta}^\lambda) d\eta' + \int_\eta^{\eta^+} \frac{H(\eta', R\phi'(\phi, \eta, \eta'))}{\sin(\phi'(\phi, \eta, \eta'))} \exp(G_{\eta, \eta'}^\lambda) d\eta' \right). \end{aligned}$$

Note the fact

$$(3.51) \quad \frac{d}{d\eta'} G_{\eta, \eta'}^\lambda(\phi) = -\frac{1 + \lambda}{\sin(\phi'(\phi, \eta, \eta'))}.$$

Hence, we can directly estimate as follows:

In Case I:

$$\begin{aligned} (3.52) \quad |f_\lambda(\eta, \phi)| &\leq \exp(-G_{\eta, 0}^\lambda) \|h\|_{L^\infty} + \|H\|_{L^\infty L^\infty} \int_0^\eta \frac{1}{\sin(\phi'(\phi, \eta, \eta'))} \exp(-G_{\eta, \eta'}^\lambda) d\eta' \\ &= \exp(-G_{\eta, 0}^\lambda) \|h\|_{L^\infty} + \frac{1}{1 + \lambda} \|H\|_{L^\infty L^\infty} \exp(-G_{\eta, \eta'}^\lambda) \Big|_0^\eta \\ &= \exp(-G_{\eta, 0}^\lambda) \|h\|_{L^\infty} + \frac{1}{1 + \lambda} \left(1 - \exp(-G_{\eta, 0}^\lambda)\right) \|H\|_{L^\infty L^\infty} \leq \|h\|_{L^\infty} + \|H\|_{L^\infty L^\infty}. \end{aligned}$$

In Case II:

$$\begin{aligned} (3.53) \quad |f_\lambda(\eta, \phi)| &\leq \exp(-G_{L, 0}^\lambda - G_{L, \eta}^\lambda) \|h\|_{L^\infty} \\ &+ \frac{1}{1 + \lambda} \left\{ \exp(-G_{L, \eta}^\lambda) \left(1 - \exp(-G_{L, 0}^\lambda)\right) \|H\|_{L^\infty L^\infty} + \left(1 - \exp(-G_{L, \eta}^\lambda)\right) \|H\|_{L^\infty L^\infty} \right\} \\ &\leq \|h\|_{L^\infty} + \|H\|_{L^\infty L^\infty}. \end{aligned}$$

In Case III:

$$\begin{aligned} (3.54) \quad |f_\lambda(\eta, \phi)| &\leq \exp(-G_{\eta^+, 0}^\lambda - G_{\eta^+, \eta}^\lambda) \|h\|_{L^\infty} \\ &+ \frac{1}{1 + \lambda} \left\{ \exp(-G_{\eta^+, \eta}^\lambda) \left(1 - \exp(-G_{\eta^+, 0}^\lambda)\right) \|H\|_{L^\infty L^\infty} + \left(1 - \exp(-G_{\eta^+, \eta}^\lambda)\right) \|H\|_{L^\infty L^\infty} \right\} \\ &\leq \|h\|_{L^\infty} + \|H\|_{L^\infty L^\infty}. \end{aligned}$$

This completes the proof of (3.41).

Step 2: Assume  $\|S\|_{L^\infty L^\infty} < \infty$  and  $\|h\|_{L^\infty} < \infty$ , then the solution  $f_\lambda^L(\eta, \phi)$  to the penalized  $\epsilon$ -Milne equation

$$(3.55) \quad \begin{cases} \lambda f_\lambda^L + \sin \phi \frac{\partial f_\lambda^L}{\partial \eta} + F(\eta) \cos \phi \frac{\partial f_\lambda^L}{\partial \phi} + f_\lambda^L - \bar{f}_\lambda^L = S(\eta, \phi), \\ f_\lambda^L(0, \phi) = h(\phi) \text{ for } \sin \phi < 0, \\ f_\lambda^L(L, \phi) = f_\lambda^L(L, R\phi). \end{cases}$$

where  $\bar{f}_\lambda^L$  is defined as (3.2), satisfies

$$(3.56) \quad \|f_\lambda^L\|_{L^\infty L^\infty} \leq \frac{1 + \lambda}{\lambda} \left( \|h\|_{L^\infty} + \|S\|_{L^\infty L^\infty} \right).$$

*The proof of (3.56):* In order to construct the solution of (3.55), we iteratively define the sequence  $\{f_m^L\}_{m=0}^\infty$  as  $f_0^L = 0$  and

$$(3.57) \quad \begin{cases} \lambda f_m^L + \sin \phi \frac{\partial f_m^L}{\partial \eta} + F(\eta) \cos \phi \frac{\partial f_m^L}{\partial \phi} + f_m^L - \bar{f}_{m-1}^L = S(\eta, \phi), \\ f_m^L(0, \phi) = h(\phi) \text{ for } \sin \phi < 0, \\ f_m^L(L, \phi) = f_m^L(L, R\phi). \end{cases}$$

Based on the analysis in Step 1 with  $H = S + \bar{f}_{m-1}^L$ , we know  $f_m^L$  is well-defined and  $\|f_m^L\|_{L^\infty L^\infty} < \infty$ . We further define  $g_m^L = f_m^L - f_{m-1}^L$  for  $m \geq 1$ . Then  $g_m^L$  can be rewritten along the characteristics as follows:

Case I:

For  $\sin \phi > 0$ ,

$$(3.58) \quad g_{m+1}^L(\eta, \phi) = \int_0^\eta \frac{\bar{g}_m^L(\eta')}{\sin(\phi'(\phi, \eta, \eta'))} \exp(-G_{\eta, \eta'}) d\eta'.$$

Case II:

For  $\sin \phi < 0$  and  $|E(\eta, \phi)| \leq e^{-V(L)}$ ,

$$(3.59) \quad g_{m+1}^L(\eta, \phi) = \left( \int_0^L \frac{\bar{g}_m^L(\eta')}{\sin(\phi'(\phi, \eta, \eta'))} \exp(-G_{L, \eta'} - G_{L, \eta}) d\eta' + \int_\eta^L \frac{\bar{g}_m^L(\eta')}{\sin(\phi'(\phi, \eta, \eta'))} \exp(G_{\eta, \eta'}) d\eta' \right).$$

Case III:

For  $\sin \phi < 0$  and  $|E(\eta, \phi)| \geq e^{-V(L)}$ ,

$$(3.60) \quad g_{m+1}^L(\eta, \phi) = \left( \int_0^{\eta^+} \frac{\bar{g}_m^L(\eta')}{\sin(\phi'(\phi, \eta, \eta'))} \exp(-G_{\eta^+, \eta'} - G_{\eta^+, \eta}) d\eta' + \int_\eta^{\eta^+} \frac{\bar{g}_m^L(\eta')}{\sin(\phi'(\phi, \eta, \eta'))} \exp(G_{\eta, \eta'}) d\eta' \right).$$

In all three cases, we can always obtain

$$(3.61) \quad \|g_{m+1}^L\|_{L^\infty L^\infty} \leq \frac{1}{1 + \lambda} \|g_m^L\|_{L^\infty L^\infty}.$$

For  $\lambda > 0$ , this is a contraction sequence. Also, we have

$$(3.62) \quad \|g_1^L\|_{L^\infty L^\infty} = \|f_1^L\|_{L^\infty L^\infty} \leq \|h\|_{L^\infty} + \|H\|_{L^\infty L^\infty}.$$

Hence,  $f_m^L$  converges strongly in  $L^\infty([0, L] \times [-\pi, \pi])$  to  $f_\lambda^L$  which satisfies (3.55). Also,  $f_\lambda^L$  satisfies

$$(3.63) \quad \|f_\lambda^L\|_{L^\infty L^\infty} \leq \frac{1 + \lambda}{\lambda} \left( \|h\|_{L^\infty} + \|S\|_{L^\infty L^\infty} \right).$$

Naturally, we obtain  $f_\lambda^L \in L^2([0, L] \times [-\pi, \pi])$ .

However, we can see the estimate blows up when  $\lambda \rightarrow 0$ . Therefore, we need to show the uniform estimate of  $f_\lambda^L$  with respect to  $\lambda$ . Similarly, we can define  $r_\lambda^L$  and  $q_\lambda^L$  for  $f_\lambda^L$  as in (3.36).

Step 3:  $r_\lambda^L$  satisfies

$$(3.64) \quad \int_0^L \|r_\lambda^L(\eta)\|_{L^2}^2 d\eta \leq 4 \|h\|_{L^2}^2 + 8 \int_0^L \|S(\eta)\|_{L^2}^2 d\eta.$$

*The proof of (3.64):* Multiplying  $f_\lambda^L$  on both sides of (3.55) and integrating over  $\phi \in [-\pi, \pi]$ , we get the energy estimate

$$(3.65) \quad \frac{1}{2} \frac{d}{d\eta} \langle f_\lambda^L, f_\lambda^L \sin \phi \rangle_\phi(\eta) = -\lambda \|f_\lambda^L(\eta)\|_{L^2}^2 - \|r_\lambda^L(\eta)\|_{L^2}^2 - F(\eta) \langle \frac{\partial f_\lambda^L}{\partial \phi}, f_\lambda^L \cos \phi \rangle_\phi(\eta) + \langle S, f_\lambda^L \rangle_\phi(\eta).$$

A further integration by parts reveals

$$(3.66) \quad -F(\eta) \langle \frac{\partial f_\lambda^L}{\partial \phi}, f_\lambda^L \cos \phi \rangle_\phi(\eta) = -\frac{1}{2} F(\eta) \langle f_\lambda^L, f_\lambda^L \sin \phi \rangle_\phi(\eta).$$

Also, the assumption  $\bar{S}(\eta) = 0$  leads to

$$(3.67) \quad \langle S, f_\lambda^L \rangle_\phi(\eta) = \langle S, q_\lambda^L \rangle_\phi(\eta) + \langle S, r_\lambda^L \rangle_\phi(\eta) = \langle S, r_\lambda^L \rangle_\phi(\eta).$$

Hence, we have the simplified form of (3.65) as follows:

$$(3.68) \quad \frac{1}{2} \frac{d}{d\eta} \langle f_\lambda^L, f_\lambda^L \sin \phi \rangle_\phi(\eta) = -\lambda \|f_\lambda^L(\eta)\|_{L^2}^2 - \|r_\lambda^L(\eta)\|_{L^2}^2 - \frac{1}{2} F(\eta) \langle f_\lambda^L, f_\lambda^L \sin \phi \rangle_\phi(\eta) + \langle S, r_\lambda^L \rangle_\phi(\eta).$$

Define

$$(3.69) \quad \alpha(\eta) = \frac{1}{2} \langle f_\lambda^L, f_\lambda^L \sin \phi \rangle_\phi(\eta).$$

Then (3.68) can be rewritten as follows:

$$(3.70) \quad \frac{d\alpha}{d\eta} = -\lambda \|f_\lambda^L(\eta)\|_{L^2}^2 - \|r_\lambda^L(\eta)\|_{L^2}^2 - F(\eta)\alpha(\eta) + \langle S, r_\lambda^L \rangle_\phi(\eta).$$

We can integrate above on  $[\eta, L]$  and  $[0, \eta]$  respectively to obtain

$$(3.71) \quad \begin{aligned} \alpha(\eta) &= \alpha(L) \exp \left( \int_\eta^L F(y) dy \right) \\ &\quad + \int_\eta^L \exp \left( \int_\eta^y F(z) dz \right) \left( \lambda \|f_\lambda^L(y)\|_{L^2}^2 + \|r_\lambda^L(y)\|_{L^2}^2 - \langle S, r_\lambda^L \rangle_\phi(y) \right) dy, \end{aligned}$$

$$(3.72) \quad \begin{aligned} \alpha(\eta) &= \alpha(0) \exp \left( - \int_0^\eta F(y) dy \right) \\ &\quad + \int_0^\eta \exp \left( - \int_y^\eta F(z) dz \right) \left( -\lambda \|f_\lambda^L(y)\|_{L^2}^2 - \|r_\lambda^L(y)\|_{L^2}^2 + \langle S, r_\lambda^L \rangle_\phi(y) \right) dy. \end{aligned}$$

The specular reflexive boundary  $f_\lambda^L(L, \phi) = f_\lambda^L(L, R\phi)$  ensures  $\alpha(L) = 0$ . Hence, based on (3.71), we have

$$(3.73) \quad \alpha(\eta) \geq \int_\eta^L \exp \left( \int_\eta^y F(z) dz \right) \left( -\langle S, r_\lambda^L \rangle_\phi(y) \right) dy.$$

Also, (3.72) implies

$$(3.74) \quad \begin{aligned} \alpha(\eta) &\leq \alpha(0) \exp[V(\eta)] + \int_0^\eta \exp \left( - \int_y^\eta F(z) dz \right) \left( \langle S, r_\lambda^L \rangle_\phi(y) \right) dy \\ &\leq 2 \|h\|_{L^2}^2 + \int_0^\eta \exp \left( - \int_y^\eta F(z) dz \right) \left( \langle S, r_\lambda^L \rangle_\phi(y) \right) dy, \end{aligned}$$

due to the fact

$$(3.75) \quad \alpha(0) = \frac{1}{2} \langle \sin \phi f_\lambda^L, f_\lambda^L \rangle_\phi(0) \leq \frac{1}{2} \left( \int_{\sin \phi > 0} h(\phi)^2 \sin \phi d\phi \right) \leq \frac{1}{2} \|h\|_{L^2}^2.$$

Then in (3.72) taking  $\eta = L$ , from  $\alpha(L) = 0$ , we have

$$(3.76) \quad \begin{aligned} \int_0^L \exp \left( \int_0^y F(z) dz \right) \|r_\lambda^L(y)\|_{L^2}^2 dy &\leq \alpha(0) + \int_0^L \exp \left( \int_0^y F(z) dz \right) \langle S, r_\lambda^L \rangle_\phi(y) dy \\ &\leq \frac{1}{2} \|h\|_{L^2}^2 + \int_0^L \exp \left( \int_0^y F(z) dz \right) \langle S, r_\lambda^L \rangle_\phi(y) dy. \end{aligned}$$

On the other hand, by (3.7), we can directly estimate as follows:

$$(3.77) \quad \int_0^L \exp \left( \int_0^y F(z) dz \right) \|r_\lambda^L(y)\|_{L^2}^2 dy \geq \frac{1}{4} \int_0^L \|r_\lambda^L(y)\|_{L^2}^2 dy.$$

Combining (3.76) and (3.77) yields

$$(3.78) \quad \int_0^L \|r_\lambda^L(\eta)\|_{L^2}^2 d\eta \leq 2 \|h\|_{L^2}^2 + 4 \int_0^L \exp \left( \int_0^y F(z) dz \right) \langle S, r_\lambda^L \rangle_\phi(y) dy.$$

By Cauchy's inequality and (3.7), we have

$$(3.79) \quad \begin{aligned} \left| \int_0^L \exp \left( \int_0^y F(z) dz \right) \langle S, r_\lambda^L \rangle_\phi(y) dy \right| &\leq \left| \int_0^L \langle S, r_\lambda^L \rangle_\phi(y) dy \right| \\ &\leq \frac{1}{8} \int_0^L \|r_\lambda^L(\eta)\|_{L^2}^2 d\eta + 2 \int_0^L \|S(\eta)\|_{L^2}^2 d\eta. \end{aligned}$$

Therefore, summarizing (3.78) and (3.79), we deduce (3.64).

Step 4:  $q_\lambda^L$  satisfies

$$(3.80) \quad \|q_\lambda^L(\eta)\|_{L^2}^2 \leq 256\pi^2(1+\lambda) \left(1 + \eta^{1/2} + \|r_\lambda^L(\eta)\|_{L^2}\right) \left(1 + \|h\|_{L^2}^2 + \int_0^L \|S(\eta)\|_{L^2}^2 d\eta + \int_0^\eta \|S(y)\|_{L^\infty} dy\right).$$

*The proof of (3.80):* Multiplying  $\sin \phi$  on both sides of (3.55) and integrating over  $\phi \in [-\pi, \pi]$  lead to

$$(3.81) \quad \frac{d}{d\eta} \langle \sin^2 \phi, f_\lambda^L \rangle_\phi(\eta) = -\lambda \langle \sin \phi, f_\lambda^L \rangle_\phi(\eta) - \langle \sin \phi, r_\lambda^L \rangle_\phi(\eta) - F(\eta) \langle \sin \phi \cos \phi, \frac{\partial f_\lambda^L}{\partial \phi} \rangle_\phi(\eta) + \langle \sin \phi, S \rangle_\phi(\eta).$$

We can further integrate by parts as follows:

$$(3.82) \quad -F(\eta) \langle \sin \phi \cos \phi, \frac{\partial f_\lambda^L}{\partial \phi} \rangle_\phi(\eta) = F(\eta) \langle \cos(2\phi), f_\lambda^L \rangle_\phi(\eta) = F(\eta) \langle \cos(2\phi), r_\lambda^L \rangle_\phi(\eta),$$

to obtain

$$(3.83) \quad \frac{d}{d\eta} \langle \sin^2 \phi, f_\lambda^L \rangle_\phi(\eta) = -\langle \sin \phi, r_\lambda^L \rangle_\phi(\eta) F(\eta) \langle \cos(2\phi), r_\lambda^L \rangle_\phi(\eta) + \langle \sin \phi, S \rangle_\phi(\eta).$$

Define

$$(3.84) \quad \beta_\lambda^L(\eta) = \langle \sin^2 \phi, f_\lambda^L \rangle_\phi(\eta).$$

Then we can simplify (3.81) as follows:

$$(3.85) \quad \frac{d\beta_\lambda^L}{d\eta} = D_\lambda^L(\eta, \phi),$$

where

$$(3.86) \quad D_\lambda^L(\eta, \phi) = -\lambda \langle \sin \phi, f_\lambda^L \rangle_\phi(\eta) - \langle \sin \phi, r_\lambda^L \rangle_\phi(\eta) + F(\eta) \langle \cos(2\phi), r_\lambda^L \rangle_\phi(\eta) + \langle \sin \phi, S \rangle_\phi(\eta).$$

Since

$$(3.87) \quad -\lambda \langle \sin \phi, f_\lambda^L \rangle_\phi(\eta) = -\lambda \langle \sin \phi, r_\lambda^L \rangle_\phi(\eta) - \lambda \langle \sin \phi, q_\lambda^L \rangle_\phi(\eta) = -\lambda \langle \sin \phi, r_\lambda^L \rangle_\phi(\eta).$$

we can further get

$$(3.88) \quad D_\lambda^L(\eta, \phi) = -\lambda \langle \sin \phi, r_\lambda^L \rangle_\phi(\eta) - \langle \sin \phi, r_\lambda^L \rangle_\phi(\eta) + F(\eta) \langle \cos(2\phi), r_\lambda^L \rangle_\phi(\eta) + \langle \sin \phi, S \rangle_\phi(\eta).$$

We can integrate over  $[0, \eta]$  in (3.85) to obtain

$$(3.89) \quad \beta_\lambda^L(\eta) = \beta_\lambda^L(0) + \int_0^\eta D_\lambda^L(y) dy.$$

It is important to note that  $D_\lambda^L$  only depends on  $r_\lambda^L$  and is independent of  $q_\lambda^L$ . Then we can directly estimate

$$(3.90) \quad \|D_\lambda^L(\eta)\|_{L^\infty} \leq 2\pi \left(1 + \lambda + |F(\eta)|\right) \|r_\lambda^L(\eta)\|_{L^2} + \|S(\eta)\|_{L^\infty} \leq 4\pi(1+\lambda) \|r_\lambda^L(\eta)\|_{L^2} + \|S(\eta)\|_{L^\infty}.$$

Also, for the initial data

$$(3.91) \quad \beta_\lambda^L(0) = \langle \sin^2 \phi, f_\lambda^L \rangle_\phi(0) \leq \left( \langle f_\lambda^L, f_\lambda^L |\sin \phi| \rangle_\phi(0) \right)^{1/2} \|\sin \phi\|_{L^2}^{3/2} \leq 8 \left( \langle f_\lambda^L, f_\lambda^L |\sin \phi| \rangle_\phi(0) \right)^{1/2}.$$

Obviously, we have

$$(3.92) \quad \langle f_\lambda^L, f_\lambda^L |\sin \phi| \rangle_\phi(0) = \int_{\sin \phi > 0} h^2(\phi) \sin \phi d\phi - \int_{\sin \phi < 0} \left( f_\lambda^L(0, \phi) \right)^2 \sin \phi d\phi.$$

However, based on the definition of  $\alpha(\eta)$  and (3.73), we can obtain

$$\begin{aligned}
 (3.93) \quad \int_{\sin \phi > 0} h^2(\phi) \sin \phi d\phi + \int_{\sin \phi < 0} \left( f_\lambda^L(0, \phi) \right)^2 \sin \phi d\phi &= 2\alpha(0) \\
 &\geq 2 \int_0^L \exp \left( \int_0^y F(z) dz \right) \left( -\langle S, r_\lambda^L \rangle_\phi(y) \right) dy \\
 &\geq -\frac{1}{2} \int_0^L \langle S, r_\lambda^L \rangle_\phi(y) dy.
 \end{aligned}$$

Hence, we can deduce

$$\begin{aligned}
 (3.94) \quad - \int_{\sin \phi < 0} \left( f_\lambda^L(0, \phi) \right)^2 \sin \phi d\phi &\leq \int_{\sin \phi > 0} h^2(\phi) \sin \phi d\phi + \frac{1}{2} \int_0^L \langle S, r_\lambda^L \rangle_\phi(y) dy \\
 &\leq \|h\|_{L^2}^2 + \frac{1}{4} \int_0^L \|r_\lambda^L(\eta)\|_{L^2}^2 d\eta + \frac{1}{4} \int_0^L \|S(\eta)\|_{L^2}^2 d\eta.
 \end{aligned}$$

From (3.64), we can deduce

$$\begin{aligned}
 (3.95) \quad \beta_\lambda^L(0)^2 &\leq 64 \langle f_\lambda^L, f_\lambda^L |\sin \phi| \rangle_\phi(0) \leq 128 \|h\|_{L^2}^2 + 16 \int_0^L \|r_\lambda^L(\eta)\|_{L^2}^2 d\eta + 16 \int_0^L \|S(\eta)\|_{L^2}^2 d\eta \\
 &\leq 192 \|h\|_{L^2}^2 + 192 \int_0^L \|S(\eta)\|_{L^2}^2 d\eta.
 \end{aligned}$$

From (3.64), (3.89), (3.90) and (3.95), we have

$$\begin{aligned}
 (3.96) \quad \|\beta_\lambda^L(\eta)\|_{L^\infty} &\leq 8 + 192 \|h\|_{L^2}^2 + 192 \int_0^L \|S(\eta)\|_{L^2}^2 d\eta + 2\pi(1+\lambda) \int_0^\eta \|r_\lambda^L(y)\|_{L^2} dy + \int_0^\eta \|S(y)\|_{L^\infty} dy \\
 &\leq 8 + 192 \|h\|_{L^2}^2 + 192 \int_0^L \|S(\eta)\|_{L^2}^2 d\eta + \int_0^\eta \|S(y)\|_{L^\infty} dy + 2\pi(1+\lambda)\eta^{1/2} \left( \int_0^\eta \|r_\lambda^L(y)\|_{L^2}^2 dy \right)^{1/2} \\
 &\leq 64\pi(1+\lambda)(1+\eta^{1/2}) \left( \|h\|_{L^2}^2 + \int_0^L \|S(\eta)\|_{L^2}^2 d\eta + \int_0^\eta \|S(y)\|_{L^\infty} dy \right).
 \end{aligned}$$

By (3.84) this implies

$$(3.97) \quad \|q_\lambda^L(\eta)\|_{L^2}^2 \leq 2 \|\beta_\lambda^L(\eta)\|_{L^2} + 2 \|r_\lambda^L(\eta)\|_{L^2} \leq 4\pi \|\beta_\lambda^L(\eta)\|_{L^\infty} + 2 \|r_\lambda^L(\eta)\|_{L^2}$$

which completes the proof of (3.80).

Step 5: Passing to the limit.

Since estimates (3.64) and (3.80) are uniform in  $\lambda$ , we can take weakly convergent subsequence  $f_\lambda^L \rightarrow f^L \in L^2([0, L] \times [-\pi, \pi])$  as  $\lambda \rightarrow 0$ . Hence,  $f^L$  is the solution of (3.34) and satisfies the estimates (3.37) and (3.38).

Step 6: Orthogonality relation (3.39).

A direct integration over  $\phi \in [-\pi, \pi]$  in (3.34) implies

$$(3.98) \quad \frac{d}{d\eta} \langle \sin \phi, f^L \rangle_\phi(\eta) = -F \langle \cos \phi, \frac{df^L}{d\phi} \rangle_\phi(\eta) + \bar{S}(\eta) = -F \langle \sin \phi, f^L \rangle_\phi(\eta).$$

thanks to  $\bar{S} = 0$ . The specular reflexive boundary  $f^L(L, \phi) = f^L(L, R\phi)$  implies  $\langle \sin \phi, f^L \rangle_\phi(L) = 0$ . Then we have

$$(3.99) \quad \langle \sin \phi, f^L \rangle_\phi(\eta) = 0.$$

It is easy to see

$$(3.100) \quad \langle \sin \phi, q^L \rangle_\phi(\eta) = 0.$$

Hence, we may derive

$$(3.101) \quad \langle \sin \phi, r^L \rangle_\phi(\eta) = 0.$$

This leads (3.39) and completes the proof of (3.3).  $\square$

3.1.2. *Infinite Slab with  $\bar{S} = 0$ .* We turn to the  $\epsilon$ -Milne problem for  $f(\eta, \phi)$  in the infinite slab  $(\eta, \phi) \in [0, \infty) \times [-\pi, \pi)$

$$(3.102) \quad \begin{cases} \sin \phi \frac{\partial f}{\partial \eta} + F(\eta) \cos \phi \frac{\partial f}{\partial \phi} + f - \bar{f} = S(\eta, \phi), \\ f(0, \phi) = h(\phi) \text{ for } \sin \phi > 0, \\ \lim_{\eta \rightarrow \infty} f(\eta, \phi) = f_\infty. \end{cases}$$

Define  $r$  and  $q$  for  $f$  as  $r^L$  and  $q^L$  for  $f^L$ .

**Lemma 3.4.** *Assume  $\bar{S}(\eta) = 0$  for any  $\eta \in [0, \infty)$  with (3.3) and (3.4). Then there exists a solution  $f(\eta, \phi)$  of the infinite slab problem (3.102), satisfying*

$$(3.103) \quad \|r\|_{L^2 L^2} \leq C \left( M + \frac{M}{K} \right)^2 < \infty,$$

$$(3.104) \quad \langle \sin \phi, r \rangle_\phi(\eta) = 0,$$

for any  $\eta \in [0, \infty)$ . Also there exists a constant  $q_\infty = f_\infty \in \mathbb{R}$  such that the following estimates hold,

$$(3.105) \quad |q_\infty| \leq C \left( 1 + M + \frac{M}{K} \right)^2 < \infty,$$

$$(3.106) \quad \|q(\eta) - q_\infty\|_{L^2} \leq C \left( \|r(\eta)\|_{L^2} + \int_\eta^\infty |F(y)| \|r(y)\|_{L^2} dy + \int_\eta^\infty \|S(y)\|_{L^\infty} dy \right),$$

$$(3.107) \quad \|q - q_\infty\|_{L^2 L^2} \leq C \left( M + \frac{M}{K} \right)^2 < \infty.$$

The solution is unique among functions such that (3.103), (3.105) and (3.107) hold.

*Proof.*

Step 1: Existence and estimates (3.103) and (3.104).

By the uniform estimates from Lemma 3.4, the solution  $f^L$  of the finite problem (3.34) in the slab  $[0, L]$  is uniformly bounded in  $L^2_{loc}([0, \infty); L^2[-\pi, \pi))$ . Then there exists a subsequence such that

$$(3.108) \quad q^L \rightharpoonup q,$$

$$(3.109) \quad r^L \rightharpoonup r,$$

weakly in  $L^2_{loc}([0, \infty); L^2[-\pi, \pi))$ . Also,  $f = q + r$  satisfies the boundary condition at  $\eta = 0$ . This shows the existence of the solution. Then property (3.103) naturally holds due to the weak lower semi-continuity of norm  $\|\cdot\|_{L^2 L^2}$ . Also, the orthogonal relation (3.104) is preserved.

Step 2: Estimates (3.105), (3.106) and (3.107).

We continue using the notation in Step 5 of the proof of Lemma 3.3. Recall (3.84) to (3.86) with  $\lambda = 0$  and  $L = \infty$ . We have

$$(3.110) \quad \beta(\eta) = \langle \sin^2 \phi, f \rangle_\phi(\eta)$$

and

$$(3.111) \quad \frac{d\beta}{d\eta} = D(\eta, \phi),$$

where

$$(3.112) \quad D(\eta, \phi) = -\langle \sin \phi, r \rangle_\phi + F(\eta) \langle \cos(2\phi), r \rangle_\phi + \langle \sin \phi, S \rangle_\phi(\eta).$$

The orthogonal relation (3.104) implies

$$(3.113) \quad D(\eta, \phi) = F(\eta) \langle \cos(2\phi), r \rangle_\phi + \langle \sin \phi, S \rangle_\phi(\eta).$$



Hence, we can integrate (3.111) over  $[0, \eta]$  to show

$$(3.114) \quad \beta(\eta) - \beta(0) = \int_0^\eta F(y) \langle \cos(2\phi), r \rangle_\phi(y) dy + \int_0^\eta \langle \sin \phi, S \rangle_\phi(y) dy.$$

Based on Lemma 3.1, since  $F \in L^1[0, \infty) \cap L^2[0, \infty)$ ,  $r \in L^2([0, \infty) \times [-\pi, \pi))$ , and  $S$  exponentially decays, by (3.95) and (3.103), there exists some constant  $\beta_\infty$  such that  $\beta_\infty = \lim_{\eta \rightarrow \infty} \beta(\eta)$  satisfying

$$(3.115) \quad |\beta_\infty| \leq |\beta(0)| + \left| \int_0^\infty F(y) \langle \cos(2\phi), r \rangle_\phi(y) dy \right| + \left| \int_0^\infty \langle \sin \phi, S \rangle_\phi(y) dy \right| \\ \leq 8 + 192 \|h\|_{L^2}^2 + 200 \int_0^\infty \|S(\eta)\|_{L^2}^2 d\eta + 2\pi \|F\|_{L^2 L^2} \|r\|_{L^2 L^2} \leq C \left( 1 + M + \frac{M}{K} \right)^2.$$

We define  $q_\infty = \beta_\infty / \|\sin \phi\|_{L^2}^2$ . Hence, the estimate of  $|q_\infty|$  in (3.105) is valid. Moreover,

$$(3.116) \quad \beta_\infty - \beta(\eta) = \int_\eta^\infty D(y) dy = \int_\eta^\infty F(y) \langle \cos(2\phi), r \rangle_\phi(y) dy + \int_\eta^\infty \langle \sin \phi, S \rangle_\phi(y) dy.$$

Note

$$(3.117) \quad \beta(\eta) = \langle \sin^2 \phi, f \rangle_\phi(\eta) = \langle \sin^2 \phi, q \rangle_\phi(\eta) + \langle \sin^2 \phi, r \rangle_\phi(\eta) = q(\eta) \|\sin \phi\|_{L^2}^2 + \langle \sin^2 \phi, r \rangle_\phi(\eta).$$

Thus we can estimate

$$(3.118) \quad \|\sin \phi\|_{L^2}^2 \|q(\eta) - q_\infty\|_{L^2} \\ = \sqrt{2\pi} \|\sin \phi\|_{L^2}^2 \|q(\eta) - q_\infty\|_{L^\infty} = \sqrt{2\pi} \left( \beta(\eta) - \langle \sin^2 \phi, r \rangle_\phi(\eta) - \beta_\infty \right) \\ \leq \sqrt{2\pi} \left( |\langle \sin^2 \phi, r \rangle_\phi(\eta)| + \int_\eta^\infty |F(y) \langle \cos(2\phi), r(y) \rangle_\phi| dy d\eta + \int_\eta^\infty |\langle \sin \phi, S \rangle_\phi(y)| dy \right) \\ \leq 2\pi^2 \left( \|r(\eta)\|_{L^2} + \int_\eta^\infty |F(y)| \|r(y)\|_{L^2} dy + \int_\eta^\infty \|S(y)\|_{L^\infty} dy \right).$$

This implies (3.106). Furthermore, we integrate (3.118) over  $\eta \in [0, \infty)$ . Cauchy's inequality and (3.9) imply

$$(3.119) \quad \int_0^\infty \left( \int_\eta^\infty |F(y)| \|r(y)\|_{L^2} dy \right)^2 d\eta \leq \|r\|_{L^2 L^2}^2 \int_0^\infty \int_\eta^\infty |F(y)|^2 dy d\eta \leq C.$$

The exponential decays shows

$$(3.120) \quad \int_0^\infty \left( \int_\eta^\infty \|S(y)\|_{L^\infty} dy \right)^2 d\eta \leq C.$$

Hence, the estimate of  $\|q - q_\infty\|_{L^2 L^2}$  in (3.107) naturally follows.

Step 3: Uniqueness

In order to show the uniqueness of the solution, we assume there are two solutions  $f_1$  and  $f_2$  to the equation (3.102) satisfying (3.103) and (3.104). Then  $f' = f_1 - f_2$  satisfies the equation

$$(3.121) \quad \begin{cases} \sin \phi \frac{\partial f'}{\partial \eta} + F(\eta) \cos \phi \frac{\partial f'}{\partial \phi} + f' - \bar{f}' = 0, \\ f'(0, \phi) = 0 \text{ for } \sin \phi > 0, \\ \lim_{\eta \rightarrow \infty} f'(\eta, \phi) = f'_\infty. \end{cases}$$

Similarly, we can define  $r'$  and  $q'$ . Multiplying  $e^{-V(\eta)} f'$  on both sides of (3.121) and integrating over  $\phi \in [-\pi, \pi)$  yields

$$(3.122) \quad \frac{1}{2} \frac{d}{d\eta} \left( \langle f', f' \sin \phi \rangle_\phi(\eta) e^{-V(\eta)} \right) = - \left( \|r'(\eta)\|_{L^2}^2 e^{-V(\eta)} \right) \leq 0.$$

This is due to the fact

$$\begin{aligned}
 (3.123) \quad & \frac{1}{2} \frac{d}{d\eta} \left( \langle f', f' \sin \phi \rangle_\phi(\eta) e^{-V(\eta)} \right) \\
 &= \left( \langle f', \frac{df'}{d\eta} \sin \phi \rangle_\phi(\eta) e^{-V(\eta)} \right) + \frac{1}{2} \left( F(\eta) \langle f', f' \sin \phi \rangle_\phi(\eta) e^{-V(\eta)} \right) \\
 &= \left( \langle f', \frac{df'}{d\eta} \sin \phi \rangle_\phi(\eta) e^{-V(\eta)} \right) + \left( F(\eta) \langle f', \frac{df'}{d\phi} \cos \phi \rangle_\phi(\eta) e^{-V(\eta)} \right).
 \end{aligned}$$

Thus, we have

$$(3.124) \quad \gamma(\eta) = \frac{1}{2} \langle f', f' \sin \phi \rangle_\phi(\eta) e^{-V(\eta)}.$$

is decreasing. Since  $r' \in L^2([0, \infty) \times [-\pi, \pi))$  and  $q' - q'_\infty \in L^2([0, \infty) \times [-\pi, \pi))$ , there exists a convergent subsequence  $\eta_k \rightarrow \infty$  satisfying  $\|r'(\eta_k)\|_{L^2} \rightarrow 0$  and  $q'(\eta_k) - q'_\infty \rightarrow 0$ . Hence, this implies

$$(3.125) \quad \frac{1}{2} \langle r', r' \sin \phi \rangle_\phi(\eta_k) e^{-V(\eta_k)} \rightarrow 0.$$

Also, due to the fact that  $q'(\eta_k)$  is independent of  $\phi$  and it is finite, we have

$$(3.126) \quad \gamma(\eta_k) \rightarrow 0.$$

By the monotonicity,  $\gamma(\eta)$  decreases to zero and  $\gamma(\eta) \geq 0$ . Then we can integrate (3.122) over  $\eta \in [0, \infty)$  to obtain

$$(3.127) \quad \gamma(\infty) - \gamma(0) = -2 \int_0^\infty \|r'(y)\|_{L^2}^2 e^{-V(y)} dy,$$

which implies

$$(3.128) \quad \gamma(0) = \langle f', f' \sin \phi \rangle_\phi(0) e^{-V(0)} = 2 \int_0^\infty \|r'(y)\|_{L^2}^2 e^{-V(y)} dy.$$

Also, we know

$$(3.129) \quad 0 \leq \frac{1}{2} \langle f', f' \sin \phi \rangle_\phi(0) e^{-V(0)} = \frac{1}{2} \langle f', f' \sin \phi \rangle_\phi(0) \leq \int_{\sin \phi > 0} (f')^2(\phi) \sin \phi d\phi = 0.$$

Naturally, we have

$$(3.130) \quad \langle f', f' \sin \phi \rangle_\phi(0) e^{-V(0)} = 2 \int_0^\infty \|r'(y)\|_{L^2}^2 e^{-V(y)} dy = 0.$$

Hence, we have  $r' = 0$  and  $f'(\eta, \phi) = q'(\eta)$ . Plugging this into the equation (3.121) reveals  $\partial_\eta q' = 0$ . Therefore,  $f'(\eta, \phi) = C$  for some constant  $C$ . Naturally the boundary data leads to  $C = 0$ . In conclusion, we must have  $f' = 0$ , which means  $f_1 = f_2$ , and the uniqueness follows.  $\square$

**3.1.3.  $\bar{S} \neq 0$  Case.** Consider the  $\epsilon$ -Milne problem for  $f(\eta, \phi)$  in  $(\eta, \phi) \in [0, \infty) \times [-\pi, \pi)$  with a general source term

$$(3.131) \quad \begin{cases} \sin \phi \frac{\partial f}{\partial \eta} + F(\eta) \cos \phi \frac{\partial f}{\partial \phi} + f - \bar{f} &= S(\eta, \phi), \\ f(0, \phi) &= h(\phi) \text{ for } \sin \phi > 0, \\ \lim_{\eta \rightarrow \infty} f(\eta, \phi) &= f_\infty. \end{cases}$$

**Lemma 3.5.** *Assume (3.3) and (3.4) hold. Then there exists a solution  $f(\eta, \phi)$  of the problem (3.131), satisfying*

$$(3.132) \quad \|r\|_{L^2 L^2} < C \left( 1 + M + \frac{M}{K} \right)^2 \leq \infty,$$

$$(3.133) \quad \langle \sin \phi, r \rangle_\phi(\eta) = - \int_\eta^\infty e^{V(\eta) - V(y)} \bar{S}(y) dy.$$

Also there exists a constant  $q_\infty = f_\infty \in \mathbb{R}$  such that the following estimates hold,

$$(3.134) \quad |q_\infty| \leq C \left(1 + M + \frac{M}{K}\right)^2 < \infty,$$

$$(3.135) \quad \|q(\eta) - q_\infty\|_{L^2} \leq C \left( \|r(\eta)\|_{L^2} + \int_\eta^\infty |F(y)| \|r(y)\|_{L^2} dy + \int_\eta^\infty \|S(y)\|_{L^\infty} dy \right),$$

$$(3.136) \quad \|q - q_\infty\|_{L^2 L^2} \leq C \left(1 + M + \frac{M}{K}\right)^2 < \infty.$$

The solution is unique among functions satisfying  $\|f - f_\infty\|_{L^2 L^2} < \infty$ .

*Proof.* We can apply superposition property for this linear problem, i.e. write  $S = \bar{S} + (S - \bar{S}) = S_Q + S_R$ . Then we solve the problem by the following steps. For simplicity, we just call the estimates (3.132), (3.134), (3.135) and (3.136) as the  $L^2$  estimates.

Step 1: Construction of auxiliary function  $f^1$ .

We first solve  $f^1$  as the solution to

$$(3.137) \quad \begin{cases} \sin \phi \frac{\partial f^1}{\partial \eta} + F(\eta) \cos \phi \frac{\partial f^1}{\partial \phi} + f^1 - \bar{f}^1 &= S_R(\eta, \phi), \\ f^1(0, \phi) &= h(\phi) \text{ for } \sin \phi > 0, \\ \lim_{\eta \rightarrow \infty} f^1(\eta, \phi) &= f_\infty^1. \end{cases}$$

Since  $\bar{S}_R = 0$ , by Lemma 3.4, we know there exists a unique solution  $f^1$  satisfying the  $L^2$  estimate.

Step 2: Construction of auxiliary function  $f^2$ .

We seek a function  $f^2$  satisfying

$$(3.138) \quad -\frac{1}{2\pi} \int_{-\pi}^{\pi} \left( \sin \phi \frac{\partial f^2}{\partial \eta} + F(\eta) \cos \phi \frac{\partial f^2}{\partial \phi} \right) d\phi + S_Q = 0.$$

The following analysis shows this type of function can always be found. An integration by parts transforms the equation (3.138) into

$$(3.139) \quad -\int_{-\pi}^{\pi} \sin \phi \frac{\partial f^2}{\partial \eta} d\phi - \int_{-\pi}^{\pi} F(\eta) \sin \phi f^2 d\phi + 2\pi S_Q = 0.$$

Setting

$$(3.140) \quad f^2(\phi, \eta) = a(\eta) \sin \phi.$$

and plugging this ansatz into (3.139), we have

$$(3.141) \quad -\frac{da}{d\eta} \int_{-\pi}^{\pi} \sin^2 \phi d\phi - F(\eta) a(\eta) \int_{-\pi}^{\pi} \sin^2 \phi d\phi + 2\pi S_Q = 0.$$

Hence, we have

$$(3.142) \quad -\frac{da}{d\eta} - F(\eta) a(\eta) + 2S_Q = 0.$$

This is a first order linear ordinary differential equation, which possesses infinite solutions. We can directly solve it to obtain

$$(3.143) \quad a(\eta) = e^{-\int_0^\eta F(y) dy} \left( a(0) + \int_0^\eta e^{\int_0^y F(z) dz} 2S_Q(y) dy \right).$$

We may take

$$(3.144) \quad a(0) = -\int_0^\infty e^{\int_0^y F(z) dz} 2S_Q(y) dy.$$

Based on the exponential decay of  $S_Q$ , we can directly verify  $a(\eta)$  decays exponentially to zero as  $\eta \rightarrow \infty$  and  $f^2$  satisfies the  $L^2$  estimate.

Step 3: Construction of auxiliary function  $f^3$ .

Based on above construction, we can directly verify

$$(3.145) \quad \int_{-\pi}^{\pi} \left( -\sin \phi \frac{\partial f^2}{\partial \eta} - F(\eta) \cos \phi \frac{\partial f^2}{\partial \phi} - f^2 + \bar{f}^2 + S_Q \right) d\phi = 0.$$

Then we can solve  $f^3$  as the solution to

$$(3.146) \quad \begin{cases} \sin \phi \frac{\partial f^3}{\partial \eta} + F(\eta) \cos \phi \frac{\partial f^3}{\partial \phi} + f^3 - \bar{f}^3 &= -\sin \phi \frac{\partial f^2}{\partial \eta} - F(\eta) \cos \phi \frac{\partial f^2}{\partial \phi} - f^2 + \bar{f}^2 + S_Q, \\ f^3(0, \phi) &= -a(0) \sin \phi \text{ for } \sin \phi > 0, \\ \lim_{\eta \rightarrow \infty} f^3(\eta, \phi) &= f_{\infty}^3. \end{cases}$$

By (3.145), we can apply Lemma 3.4 to obtain a unique solution  $f^3$  satisfying the  $L^2$  estimate.

Step 4: Construction of auxiliary function  $f^4$ .

We now define  $f^4 = f^2 + f^3$  and an explicit verification shows

$$(3.147) \quad \begin{cases} \sin \phi \frac{\partial f^4}{\partial \eta} + F(\eta) \cos \phi \frac{\partial f^4}{\partial \phi} + f^4 - \bar{f}^4 &= S_Q(\eta, \phi), \\ f^4(0, \phi) &= 0 \text{ for } \sin \phi > 0, \\ \lim_{\eta \rightarrow \infty} f^4(\eta, \phi) &= f_{\infty}^4, \end{cases}$$

and  $f^4$  satisfies the  $L^2$  estimate.

In summary, we deduce that  $f^1 + f^4$  is the solution of (3.131) and satisfies the  $L^2$  estimate. A direct computation of  $\langle \sin \phi, f^i \rangle_{\phi}(\eta)$  for  $i = 1, 2, 3, 4$  leads to (3.133). From  $\|f - f_{\infty}\|_{L^2 L^2} < \infty$ , we deduce  $\|\bar{f} - f_{\infty}\|_{L^2 L^2} < \infty$ , a similar argument as in Lemma 3.4 shows the uniqueness of solution.  $\square$

Combining all above, we have the following theorem.

**Theorem 3.6.** *For the  $\epsilon$ -Milne problem (3.1), there exists a unique solution  $f(\eta, \phi)$  satisfying the estimates*

$$(3.148) \quad \|f - f_{\infty}\|_{L^2 L^2} \leq C \left( 1 + M + \frac{M}{K} \right) < \infty,$$

for some real number  $f_{\infty}$  satisfying

$$(3.149) \quad |f_{\infty}| \leq C \left( 1 + M + \frac{M}{K} \right)^2 < \infty.$$

### 3.2. $L^{\infty}$ Estimates.

3.2.1. *Finite Slab.* Consider the  $\epsilon$ -transport problem for  $f(\eta, \phi)$  in a finite slab  $(\eta, \phi) \in [0, L] \times [-\pi, \pi)$

$$(3.150) \quad \begin{cases} \sin \phi \frac{\partial f}{\partial \eta} + F(\eta) \cos \phi \frac{\partial f}{\partial \phi} + f &= H(\eta, \phi), \\ f(0, \phi) &= h(\phi) \text{ for } \sin \phi > 0, \\ f(L, \phi) &= f(L, R\phi). \end{cases}$$

Define the energy as follows:

$$(3.151) \quad E(\eta, \phi) = \cos \phi e^{-V(\eta)}.$$

In the plane  $(\eta, \phi) \in [0, \infty) \times [-\pi, \pi)$ , on the curve  $\phi = \phi(\eta)$  with constant energy, we can see

$$(3.152) \quad \frac{dE}{d\eta} = \frac{\partial E}{\partial \eta} + \frac{\partial E}{\partial \phi} \frac{\partial \phi}{\partial \eta} = \cos \phi F(\eta) e^{-V(\eta)} - \sin \phi e^{-V(\eta)} \frac{\partial \phi}{\partial \eta} = 0,$$

which further implies

$$(3.153) \quad \frac{\partial \phi}{\partial \eta} = \frac{\cos \phi F(\eta)}{\sin \phi}.$$

Plugging this into the equation (3.150), on this curve, we deduce

$$(3.154) \quad \frac{df}{d\eta} = \frac{\partial f}{\partial \eta} + \frac{\partial f}{\partial \phi} \frac{\partial \phi}{\partial \eta} = \frac{1}{\sin \phi} \left( \sin \phi \frac{\partial f}{\partial \eta} + \cos \phi F(\eta) \frac{\partial f}{\partial \phi} \right).$$

Hence, this curve with constant energy is exactly the characteristics of the equation (3.150). Also, on this curve the equation can be simplified as follows:

$$(3.155) \quad \sin \phi \frac{df}{d\eta} + f = H.$$

An implicit function  $\eta^+(\eta, \phi)$  can be determined through

$$(3.156) \quad |E(\eta, \phi)| = e^{-V(\eta^+)}.$$

which means  $(\eta^+, \phi_0)$  with  $\sin \phi_0 = 0$  is on the same characteristics as  $(\eta, \phi)$ . Define the quantities for  $0 \leq \eta' \leq \eta^+$  as follows:

$$(3.157) \quad \phi'(\phi, \eta, \eta') = \cos^{-1}(\cos \phi e^{V(\eta') - V(\eta)}),$$

$$(3.158) \quad R\phi'(\phi, \eta, \eta') = -\cos^{-1}(\cos \phi e^{V(\eta') - V(\eta)}) = -\phi'(\phi, \eta, \eta'),$$

where the inverse trigonometric function can be defined single-valued in the domain  $[0, \pi)$  and the quantities are always well-defined due to the monotonicity of  $V$ . Finally we put

$$(3.159) \quad G_{\eta, \eta'}(\phi) = \int_{\eta'}^{\eta} \frac{1}{\sin(\phi'(\phi, \eta, \xi))} d\xi.$$

We can rewrite the solution to the equation (3.150) along the characteristics as follows:

Case I:

For  $\sin \phi > 0$ ,

$$(3.160) \quad f(\eta, \phi) = h(\phi'(\phi, \eta, 0)) \exp(-G_{\eta, 0}) + \int_0^{\eta} \frac{H(\eta', \phi'(\phi, \eta, \eta'))}{\sin(\phi'(\phi, \eta, \eta'))} \exp(-G_{\eta, \eta'}) d\eta'.$$

Case II:

For  $\sin \phi < 0$  and  $|E(\eta, \phi)| \leq e^{-V(L)}$ ,

$$(3.161) \quad \begin{aligned} f(\eta, \phi) &= h(\phi'(\phi, \eta, 0)) \exp(-G_{L, 0} - G_{L, \eta}) \\ &+ \left( \int_0^L \frac{H(\eta', \phi'(\phi, \eta, \eta'))}{\sin(\phi'(\phi, \eta, \eta'))} \exp(-G_{L, \eta'} - G_{L, \eta}) d\eta' + \int_{\eta}^L \frac{H(\eta', R\phi'(\phi, \eta, \eta'))}{\sin(\phi'(\phi, \eta, \eta'))} \exp(G_{\eta, \eta'}) d\eta' \right). \end{aligned}$$

Case III:

For  $\sin \phi < 0$  and  $|E(\eta, \phi)| \geq e^{-V(L)}$ ,

$$(3.162) \quad \begin{aligned} f(\eta, \phi) &= h(\phi'(\phi, \eta, 0)) \exp(-G_{\eta^+, 0} - G_{\eta^+, \eta}) \\ &+ \left( \int_0^{\eta^+} \frac{H(\eta', \phi'(\phi, \eta, \eta'))}{\sin(\phi'(\phi, \eta, \eta'))} \exp(-G_{\eta^+, \eta'} - G_{\eta^+, \eta}) d\eta' + \int_{\eta}^{\eta^+} \frac{H(\eta', R\phi'(\phi, \eta, \eta'))}{\sin(\phi'(\phi, \eta, \eta'))} \exp(G_{\eta, \eta'}) d\eta' \right). \end{aligned}$$

3.2.2. *Infinite Slab.* Consider the  $\epsilon$ -transport problem for  $f(\eta, \phi)$  in an infinite slab  $(\eta, \phi) \in [0, \infty) \times [-\pi, \pi)$

$$(3.163) \quad \begin{cases} \sin \phi \frac{\partial f}{\partial \eta} + F(\eta) \cos \phi \frac{\partial f}{\partial \phi} + f = H(\eta, \phi), \\ f(0, \phi) = h(\phi) \text{ for } \sin \phi > 0, \\ \lim_{\eta \rightarrow \infty} f(\eta, \phi) = f_{\infty}. \end{cases}$$

We can define the solution via taking limit  $L \rightarrow \infty$  in (3.160), (3.161) and (3.162) as follows:

$$(3.164) \quad f(\eta, \phi) = \mathcal{A}h(\phi) + \mathcal{T}H(\eta, \phi),$$

where

Case I:

For  $\sin \phi > 0$ ,

$$(3.165) \quad \mathcal{A}h(\phi) = h(\phi'(\phi, \eta, 0)) \exp(-G_{\eta,0})$$

$$(3.166) \quad \mathcal{T}H(\eta, \phi) = \int_0^\eta \frac{H(\eta', \phi'(\phi, \eta, \eta'))}{\sin(\phi'(\phi, \eta, \eta'))} \exp(-G_{\eta, \eta'}) d\eta'.$$

Case II:

For  $\sin \phi < 0$  and  $|E(\eta, \phi)| \leq e^{-V_\infty}$ ,

$$(3.167) \quad \mathcal{A}h(\phi) = 0$$

$$(3.168) \quad \mathcal{T}H(\eta, \phi) = \int_\eta^\infty \frac{H(\eta', R\phi'(\phi, \eta, \eta'))}{\sin(\phi'(\phi, \eta, \eta'))} \exp(G_{\eta, \eta'}) d\eta'.$$

Case III:

For  $\sin \phi < 0$  and  $|E(\eta, \phi)| \geq e^{-V_\infty}$ ,

$$(3.169) \quad \mathcal{A}h(\phi) = h(\phi'(\phi, \eta, 0)) \exp(-G_{\eta^+,0} - G_{\eta^+, \eta})$$

$$(3.170) \quad \begin{aligned} \mathcal{T}H(\eta, \phi) = & \left( \int_0^{\eta^+} \frac{H(\eta', \phi'(\phi, \eta, \eta'))}{\sin(\phi'(\phi, \eta, \eta'))} \exp(-G_{\eta^+, \eta'} - G_{\eta^+, \eta}) d\eta' \right. \\ & \left. + \int_\eta^{\eta^+} \frac{H(\eta', R\phi'(\phi, \eta, \eta'))}{\sin(\phi'(\phi, \eta, \eta'))} \exp(G_{\eta, \eta'}) d\eta' \right). \end{aligned}$$

Notice that

$$(3.171) \quad \lim_{L \rightarrow \infty} \exp(-G_{L, \eta}) = 0,$$

for  $\sin \phi < 0$  and  $|E(\eta, \phi)| \leq e^{-V_\infty}$ . Hence, above derivation is valid. In order to achieve the estimate of  $f$ , we need to control  $\mathcal{T}H$  and  $\mathcal{A}h$ .

3.2.3. *Preliminaries.* We first give several technical lemmas to be used for proving  $L^\infty$  estimates of  $f$ .

**Lemma 3.7.** *For any  $0 \leq \beta \leq 1$ , we have*

$$(3.172) \quad \|e^{\beta\eta} \mathcal{A}h\|_{L^\infty} \leq \|h\|_{L^\infty}.$$

*In particular,*

$$(3.173) \quad \|\mathcal{A}h\|_{L^\infty} \leq \|h\|_{L^\infty}.$$

*Proof.* Since  $\phi'$  is always in the domain  $[0, \pi)$ , we naturally have

$$(3.174) \quad 0 \leq \sin(\phi'(\phi, \eta, \xi)) \leq 1,$$

which further implies

$$(3.175) \quad \frac{1}{\sin(\phi'(\phi, \eta, \xi))} \geq 1.$$

Combined with the fact  $\eta^+ \geq \eta$ , we deduce

$$(3.176) \quad \exp(-G_{\eta,0}) \leq e^{-\eta}$$

$$(3.177) \quad \exp(-G_{\eta^+,0} - G_{\eta^+, \eta}) \leq \exp(-G_{\eta^+,0}) \leq \exp(-G_{\eta,0}) \leq e^{-\eta}.$$

Hence, our result easily follows. □

**Lemma 3.8.** *The integral operator  $\mathcal{T}$  satisfies*

$$(3.178) \quad \|\mathcal{T}H\|_{L^\infty L^\infty} \leq C \|H\|_{L^\infty L^\infty},$$

*and for any  $0 \leq \beta \leq 1/2$*

$$(3.179) \quad \|e^{\beta\eta} \mathcal{T}H\|_{L^\infty L^\infty} \leq C \|e^{\beta\eta} H\|_{L^\infty L^\infty},$$

*where  $C$  is a universal constant independent of data.*

*Proof.* For (3.178), when  $\sin \phi > 0$

$$(3.180) \quad \begin{aligned} |\mathcal{T}H| &\leq \int_0^\eta |H(\eta', \phi'(\phi, \eta, \eta'))| \frac{1}{\sin(\phi'(\phi, \eta, \eta'))} \exp(-G_{\eta, \eta'}) d\eta' \\ &\leq \|H\|_{L^\infty L^\infty} \int_0^\eta \frac{1}{\sin(\phi'(\phi, \eta, \eta'))} \exp(-G_{\eta, \eta'}) d\eta'. \end{aligned}$$

We can directly estimate

$$(3.181) \quad \int_0^\eta \frac{1}{\sin(\phi'(\phi, \eta, \eta'))} \exp(-G_{\eta, \eta'}) d\eta' \leq \int_0^\infty e^{-z} dz = 1,$$

and (3.178) naturally follows. For  $\sin \phi < 0$  and  $|E(\eta, \phi)| \leq e^{-V_\infty}$ ,

$$(3.182) \quad \begin{aligned} |\mathcal{T}H| &\leq \int_\eta^\infty |H(\eta', \phi'(\phi, \eta, \eta'))| \frac{1}{\sin(\phi'(\phi, \eta, \eta'))} \exp(G_{\eta, \eta'}) d\eta' \\ &\leq \|H\|_{L^\infty L^\infty} \int_\eta^\infty \frac{1}{\sin(\phi'(\phi, \eta, \eta'))} \exp(G_{\eta, \eta'}) d\eta'. \end{aligned}$$

we have

$$(3.183) \quad \int_\eta^\infty \frac{1}{\sin(\phi'(\phi, \eta, \eta'))} \exp(G_{\eta, \eta'}) d\eta' \leq \int_{-\infty}^0 e^z dz = 1,$$

and (3.178) easily follows. The case  $\sin \phi < 0$  and  $|E(\eta, \phi)| \geq e^{-V_\infty}$  can be proved combining above two techniques, so we omit it here.

For (3.179), when  $\sin \phi > 0$ ,  $\eta \geq \eta'$  and  $\beta < 1/2$ , since  $G_{\eta, \eta'} \geq \eta - \eta'$ , we have

$$(3.184) \quad \beta(\eta - \eta') - G_{\eta, \eta'} \leq \beta(\eta - \eta') - \frac{1}{2}(\eta - \eta') - \frac{1}{2}G_{\eta, \eta'} \leq -\frac{1}{2}G_{\eta, \eta'}.$$

Then it is natural that

$$(3.185) \quad \begin{aligned} \int_0^\eta \frac{1}{\sin(\phi'(\phi, \eta, \eta'))} \exp(\beta(\eta - \eta') - G_{\eta, \eta'}) d\eta' &\leq \int_0^\eta \frac{1}{\sin(\phi'(\phi, \eta, \eta'))} \exp(-G_{\eta, \eta'}/2) d\eta' \\ &\leq \int_0^\infty e^{-z/2} dz = 2. \end{aligned}$$

This leads to

$$(3.186) \quad \begin{aligned} |e^{\beta\eta} \mathcal{T}H| &\leq e^{\beta\eta} \int_0^\eta |H(\eta', \phi'(\phi, \eta, \eta'))| \frac{1}{\sin(\phi'(\phi, \eta, \eta'))} \exp(-G_{\eta, \eta'}) d\eta' \\ &\leq \|e^{\beta\eta} H\|_{L^\infty L^\infty} \int_0^\eta \frac{1}{\sin(\phi'(\phi, \eta, \eta'))} \exp(\beta(\eta - \eta') - G_{\eta, \eta'}) d\eta' \\ &\leq C \|e^{\beta\eta} H\|_{L^\infty L^\infty}, \end{aligned}$$

and (3.179) naturally follows. For  $\sin \phi < 0$  and  $|E(\eta, \phi)| \leq e^{-V_\infty}$ , note for  $\eta' \geq \eta$

$$(3.187) \quad \beta(\eta - \eta') + G_{\eta, \eta'} \leq \beta(\eta - \eta') + \frac{1}{2}(\eta - \eta') + \frac{1}{2}G_{\eta, \eta'} \leq \frac{1}{2}G_{\eta, \eta'}.$$

Then (3.179) holds by obvious modifications of  $\sin \phi > 0$  case. The case  $\sin \phi < 0$  and  $|E(\eta, \phi)| \geq e^{-V_\infty}$  can be shown by combining above two cases, so we omit it here.  $\square$

**Lemma 3.9.** *For any  $\delta > 0$  there is a constant  $C(\delta) > 0$  independent of data such that*

$$(3.188) \quad \|\mathcal{T}H\|_{L^\infty L^2} \leq C(\delta) \|H\|_{L^2 L^2} + \delta \|H\|_{L^\infty L^\infty}.$$

*Proof.* We divide the proof into several steps.

Step 1: The case of  $\sin \phi > 0$ .

We consider

$$\begin{aligned}
 (3.189) \quad \int_{\sin \phi > 0} |\mathcal{T}H(\eta)|^2 d\phi &= \int_{\sin \phi > 0} \left( \int_0^\eta \frac{H(\eta', \phi'(\phi, \eta, \eta'))}{\sin(\phi'(\phi, \eta, \eta'))} \exp(-G_{\eta, \eta'}) d\eta' \right)^2 d\phi \\
 &= \int \left( \int \mathbf{1}_{\{\sin(\phi'(\phi, \eta, \eta')) > m\}} \cdots \right)^2 + \int \left( \int \mathbf{1}_{\{\sin(\phi'(\phi, \eta, \eta')) \leq m\}} \cdots \right)^2 \\
 &= I_1 + I_2.
 \end{aligned}$$

By Cauchy's inequality and (3.181), we get

$$\begin{aligned}
 (3.190) \quad I_1 &\leq \int_{\sin \phi > 0} \left( \int_0^\eta |H(\eta', \phi'(\phi, \eta, \eta'))|^2 d\eta' \right) \left( \int_0^\eta \mathbf{1}_{\{\sin(\phi'(\phi, \eta, \eta')) > m\}} \frac{\exp(-2G_{\eta, \eta'})}{\sin^2(\phi'(\phi, \eta, \eta'))} d\eta' \right) d\phi \\
 &\leq \frac{1}{m} \|H\|_{L^2 L^2}^2 \int_{\sin \phi > 0} \left( \int_0^\eta \mathbf{1}_{\{\sin(\phi'(\phi, \eta, \eta')) > m\}} \frac{\exp(-2G_{\eta, \eta'})}{\sin(\phi'(\phi, \eta, \eta'))} d\eta' \right) d\phi \\
 &\leq \frac{\pi}{m} \|H\|_{L^2 L^2}^2.
 \end{aligned}$$

On the other hand, for  $\eta' \leq \eta$ , we can directly estimate  $\phi'(\phi, \eta, \eta') \geq \phi$ . Hence, we have the relation

$$(3.191) \quad \sin \phi \leq \sin(\phi'(\phi, \eta, \eta')).$$

Therefore, we can directly estimate  $I_2$  as follows:

$$\begin{aligned}
 (3.192) \quad I_2 &\leq \|H\|_{L^\infty L^\infty}^2 \int_{\sin \phi > 0} \left( \int_0^\eta \mathbf{1}_{\{\sin(\phi'(\phi, \eta, \eta')) \leq m\}} \frac{1}{\sin(\phi'(\phi, \eta, \eta'))} \exp(-G_{\eta, \eta'}) d\eta' \right)^2 d\phi \\
 &\leq \|H\|_{L^\infty L^\infty}^2 \int_{\sin \phi > 0} \left( \int_0^\eta \mathbf{1}_{\{\sin \phi \leq m\}} \frac{1}{\sin(\phi'(\phi, \eta, \eta'))} \exp(-G_{\eta, \eta'}) d\eta' \right)^2 d\phi \\
 &= \|H\|_{L^\infty L^\infty}^2 \int_{\sin \phi > 0} \mathbf{1}_{\{\sin \phi \leq m\}} \left( \int_0^\eta \frac{1}{\sin(\phi'(\phi, \eta, \eta'))} \exp(-G_{\eta, \eta'}) d\eta' \right)^2 d\phi.
 \end{aligned}$$

Note

$$(3.193) \quad \int_0^\eta \frac{1}{\sin(\phi'(\phi, \eta, \eta'))} \exp(-G_{\eta, \eta'}) d\eta' \leq \int_0^\infty e^{-z} dz = 1.$$

Therefore, for  $m$  sufficiently small, we have

$$(3.194) \quad I_2 \leq \|H\|_{L^\infty L^\infty}^2 \int_{\sin \phi > 0} \mathbf{1}_{\{\sin \phi \leq m\}} d\phi \leq 4m.$$

Summing up (3.190) and (3.194), for  $m$  sufficiently small, we deduce (3.188).

Step 2: The case of  $\sin \phi < 0$  and  $|E(\eta, \phi)| \leq e^{-V_\infty}$ .

We can decompose

$$\begin{aligned}
 (3.195) \quad &\int_{\sin \phi < 0} \mathbf{1}_{\{|E(\eta, \phi)| \leq e^{-V_\infty}\}} |\mathcal{T}H|^2 d\phi \\
 &= \int_{\sin \phi < 0} \mathbf{1}_{\{|E(\eta, \phi)| \leq e^{-V_\infty}\}} \left( \int_\eta^\infty \frac{H(\eta', R\phi'(\phi, \eta, \eta'))}{\sin(\phi'(\phi, \eta, \eta'))} \exp(G_{\eta, \eta'}) d\eta' \right)^2 d\phi \\
 &= \int \left( \int \mathbf{1}_{\{\sin(\phi'(\phi, \eta, \eta')) > m\}} \cdots \right)^2 + \int \left( \int \mathbf{1}_{\{\sin(\phi'(\phi, \eta, \eta')) \leq m\}} \mathbf{1}_{\{\eta' - \eta \geq \sigma\}} \cdots \right)^2 \\
 &\quad \int \left( \int \mathbf{1}_{\{\sin(\phi'(\phi, \eta, \eta')) \leq m\}} \mathbf{1}_{\{\eta' - \eta \leq \sigma\}} \cdots \right)^2 \\
 &= I_1 + I_2 + I_3.
 \end{aligned}$$



We can directly estimate  $I_1$  as follows:

$$\begin{aligned}
 (3.196) \quad I_1 &\leq \int_{\sin \phi < 0} \left( \int_{\eta}^{\infty} |H(\eta', R\phi'(\phi, \eta, \eta'))|^2 d\eta' \right) \left( \int_{\eta}^{\infty} \mathbf{1}_{\{\sin(\phi'(\phi, \eta, \eta')) > m\}} \frac{\exp(2G_{\eta, \eta'})}{\sin^2(\phi'(\phi, \eta, \eta'))} d\eta' \right) d\phi \\
 &\leq \frac{1}{m} \|H\|_{L^2 L^2}^2 \int_{\sin \phi < 0} \left( \int_{\eta}^{\infty} \mathbf{1}_{\{\sin(\phi'(\phi, \eta, \eta')) > m\}} \frac{\exp(2G_{\eta, \eta'})}{\sin(\phi'(\phi, \eta, \eta'))} d\eta' \right) d\phi \\
 &\leq \frac{\pi}{m} \|H\|_{L^2 L^2}^2.
 \end{aligned}$$

On the other hand, for  $I_2$  we have

$$(3.197) \quad I_2 \leq \|H\|_{L^\infty L^\infty}^2 \int_{\sin \phi < 0} \mathbf{1}_{\{|E(\eta, \phi)| \leq e^{-V_\infty}\}} \left( \int_{\eta}^{\infty} \mathbf{1}_{\{\sin(\phi'(\phi, \eta, \eta')) \leq m\}} \mathbf{1}_{\{\eta' - \eta \geq \sigma\}} \frac{\exp(G_{\eta, \eta'})}{\sin(\phi'(\phi, \eta, \eta'))} d\eta' \right)^2 d\phi.$$

Note

$$(3.198) \quad G_{\eta, \eta'} = \int_{\eta'}^{\eta} \frac{1}{\sin(\phi'(\phi, \eta, y))} dy \leq -\frac{\eta' - \eta}{m} = -\frac{\sigma}{m}.$$

Then we can obtain

$$(3.199) \quad I_2 \leq \|H\|_{L^\infty L^\infty}^2 \int_{\sin \phi < 0} \left( \int_{\infty}^{-\sigma/m} e^z dz \right)^2 d\phi = 4e^{-\frac{\sigma}{m}} \|H\|_{L^\infty L^\infty}^2.$$

For  $I_3$ , we can estimate as follows:

$$\begin{aligned}
 (3.200) \quad I_3 &\leq \|H\|_{L^\infty L^\infty}^2 \int_{\sin \phi < 0} \mathbf{1}_{\{|E(\eta, \phi)| \leq e^{-V_\infty}\}} \left( \int_{\eta}^{\infty} \mathbf{1}_{\{\sin(\phi'(\phi, \eta, \eta')) \leq m\}} \mathbf{1}_{\{\eta' - \eta \leq \sigma\}} \frac{\exp(G_{\eta, \eta'})}{\sin(\phi'(\phi, \eta, \eta'))} d\eta' \right)^2 d\phi \\
 &\leq \|H\|_{L^\infty L^\infty}^2 \int_{\sin \phi < 0} \mathbf{1}_{\{|E(\eta, \phi)| \leq e^{-V_\infty}\}} \left( \int_{\eta}^{\eta + \sigma} \mathbf{1}_{\{\sin(\phi'(\phi, \eta, \eta')) \leq m\}} \mathbf{1}_{\{\eta' - \eta \leq \sigma\}} \frac{\exp(G_{\eta, \eta'})}{\sin(\phi'(\phi, \eta, \eta'))} d\eta' \right)^2 d\phi.
 \end{aligned}$$

Note

$$(3.201) \quad \int_{\eta}^{\infty} \frac{1}{\sin(\phi'(\phi, \eta, \eta'))} \exp(G_{\eta, \eta'}) d\eta' \leq \int_{-\infty}^0 e^z dz = 1.$$

Then  $1 \leq \alpha = e^{V(\eta') - V(\eta)} \leq e^{V(\eta + \sigma) - V(\eta)} \leq 1 + 4\sigma$  due to (3.8), and for  $\eta' \in [\eta, \eta + \sigma]$ ,  $\sin \phi'(\phi, \eta, \eta') = \sin \left( \cos^{-1}(\alpha \cos \phi) \right)$ ,  $\sin(\phi'(\phi, \eta, \eta')) < m$  lead to

$$\begin{aligned}
 (3.202) \quad |\sin \phi| &= \sqrt{1 - \cos^2 \phi} = \sqrt{1 - \frac{\cos^2 \phi'(\phi, \eta, \eta')}{\alpha^2}} = \frac{\sqrt{\alpha^2 - (1 - \sin^2 \phi'(\phi, \eta, \eta'))}}{\alpha} \\
 &\leq \frac{\sqrt{\alpha^2 - 1 + m^2}}{\alpha} \leq \frac{\sqrt{(1 + 4\sigma)^2 - 1 + m^2}}{\alpha} \leq \sqrt{9\sigma + m^2}.
 \end{aligned}$$

Hence, we can obtain

$$\begin{aligned}
 (3.203) \quad I_3 &\leq \|H\|_{L^\infty L^\infty}^2 \int_{\sin \phi < 0} \mathbf{1}_{\{\sin(\phi'(\phi, \eta, \eta')) \leq m\}} d\phi \leq \|H\|_{L^\infty L^\infty}^2 \int_{\sin \phi < 0} \mathbf{1}_{\{|\sin \phi| \leq \sqrt{9\sigma + m^2}\}} d\phi \\
 &\leq 4\sqrt{9\sigma + m^2}.
 \end{aligned}$$

Summarizing (3.196), (3.199) and (3.203), for sufficiently small  $\sigma$ , we can always choose  $m$  small enough to guarantee the relation (3.188).

Step 3: The case of  $\sin \phi < 0$  and  $|E(\eta, \phi)| \geq e^{-V_\infty}$ .

We can decompose  $\mathcal{TH}$  as follows:

$$\begin{aligned}
 (3.204) \quad & \mathcal{TH}(\eta, \phi) \\
 &= \left( \int_0^{\eta^+} \dots \exp(-G_{\eta^+, \eta'} - G_{\eta^+, \eta}) d\eta' + \int_\eta^{\eta^+} \dots \exp(G_{\eta, \eta'}) d\eta' \right) \\
 &= \int_0^\eta \dots \exp(-G_{\eta^+, \eta'} - G_{\eta^+, \eta}) d\eta' + \left( \int_\eta^{\eta^+} \dots \exp(-G_{\eta^+, \eta'} - G_{\eta^+, \eta}) d\eta' + \int_\eta^{\eta^+} \dots \exp(G_{\eta, \eta'}) d\eta' \right) \\
 &= I_1 + I_2.
 \end{aligned}$$

For  $I_1$ , we can apply a similar argument as in Step 1 and for  $I_2$ , a similar argument as in Step 2 conclude the lemma.  $\square$

3.2.4. *Estimates of  $\epsilon$ -Milne Equation.* Consider the equation satisfied by  $z = f - f_\infty$  as follows:

$$(3.205) \quad \begin{cases} \sin \phi \frac{\partial z}{\partial \eta} + F(\eta) \cos \phi \frac{\partial z}{\partial \phi} + z &= \bar{z} + S, \\ z(0, \phi) &= p(\phi) = h(\phi) - f_\infty \text{ for } \sin \phi > 0, \\ \lim_{\eta \rightarrow \infty} z(\eta, \phi) &= 0. \end{cases}$$

**Lemma 3.10.** *Assume (3.3) and (3.4) hold. Then there exists a constant  $C$  depending on the data such that the solution of equation (3.205) verifies*

$$(3.206) \quad \|z\|_{L^\infty L^\infty} \leq C \left( 1 + M + \frac{M}{K} + \|z\|_{L^2 L^2} \right).$$

*Proof.* We first show the following important facts:

$$(3.207) \quad \|\bar{z}\|_{L^2 L^2} \leq \|z\|_{L^2 L^2},$$

$$(3.208) \quad \|\bar{z}\|_{L^\infty L^\infty} \leq \|z\|_{L^\infty L^2}.$$

We can directly derive them by Cauchy's inequality as follows:

$$\begin{aligned}
 (3.209) \quad \|\bar{z}\|_{L^2 L^2}^2 &= \int_0^\infty \int_{-\pi}^\pi \left( \frac{1}{2\pi} \right)^2 \left( \int_{-\pi}^\pi z(\eta, \phi) d\phi \right)^2 d\phi d\eta \leq \int_0^\infty \int_{-\pi}^\pi \left( \frac{1}{2\pi} \right) \left( \int_{-\pi}^\pi z^2(\eta, \phi) d\phi \right) d\phi d\eta \\
 &= \int_0^\infty \left( \int_{-\pi}^\pi z^2(\eta, \phi) d\phi \right) d\eta = \|z\|_{L^2 L^2}^2.
 \end{aligned}$$

$$\begin{aligned}
 (3.210) \quad \|\bar{z}\|_{L^\infty L^\infty}^2 &= \sup_\eta \bar{z}^2(\eta) = \sup_\eta \left( \frac{1}{2\pi} \int_{-\pi}^\pi z(\eta, \phi) d\phi \right)^2 \leq \sup_\eta \left( \frac{1}{2\pi} \right)^2 \left( \int_{-\pi}^\pi z^2(\eta, \phi) d\phi \right) \left( \int_{-\pi}^\pi 1^2 d\phi \right) \\
 &= \sup_\eta \left( \int_{-\pi}^\pi z^2(\eta, \phi) d\phi \right) = \|z\|_{L^\infty L^2}^2.
 \end{aligned}$$

By (3.205),  $z = \mathcal{A}p + \mathcal{T}(\bar{z} + S)$  leads to

$$(3.211) \quad \mathcal{T}(\bar{z} + S) = z - \mathcal{A}p,$$

Then by Lemma 3.9, (3.207) and (3.208), we can show

$$\begin{aligned}
 (3.212) \quad \|z - \mathcal{A}p\|_{L^\infty L^2} &\leq C(\delta) \left( \|\bar{z}\|_{L^2 L^2} + \|S\|_{L^2 L^2} \right) + \delta \left( \|\bar{z}\|_{L^\infty L^\infty} + \|S\|_{L^\infty L^\infty} \right) \\
 &\leq C(\delta) \left( \|z\|_{L^2 L^2} + \|S\|_{L^2 L^2} \right) + \delta \left( \|z\|_{L^\infty L^2} + \|S\|_{L^\infty L^\infty} \right).
 \end{aligned}$$

Therefore, based on Lemma 3.7 and (3.212), we can directly estimate

$$(3.213) \quad \begin{aligned} \|z\|_{L^\infty L^2} &\leq \|Ap\|_{L^\infty} + C(\delta) \left( \|z\|_{L^2 L^2} + \|S\|_{L^2 L^2} \right) + \delta \left( \|z\|_{L^\infty L^2} + \|S\|_{L^\infty L^\infty} \right) \\ &\leq \|p\|_{L^\infty} + C(\delta) \left( \|z\|_{L^2 L^2} + \|S\|_{L^2 L^2} \right) + \delta \left( \|z\|_{L^\infty L^2} + \|S\|_{L^\infty L^\infty} \right). \end{aligned}$$

We can take  $\delta = 1/2$  to obtain

$$(3.214) \quad \|z\|_{L^\infty L^2} \leq C \left( \|z\|_{L^2 L^2} + \|S\|_{L^2 L^2} + \|S\|_{L^\infty L^\infty} + \|p\|_{L^\infty} \right).$$

Therefore, based on Lemma 3.8, (3.214) and (3.208), we can achieve

$$(3.215) \quad \begin{aligned} \|z\|_{L^\infty L^\infty} &\leq \|Ap\|_{L^\infty L^\infty} + \|\mathcal{T}(\bar{z} + S)\|_{L^\infty L^\infty} \leq C \left( \|p\|_{L^\infty} + \|\bar{z}\|_{L^\infty L^\infty} + \|S\|_{L^\infty L^\infty} \right) \\ &\leq C \left( \|p\|_{L^\infty} + \|S\|_{L^\infty L^\infty} + \|z\|_{L^\infty L^2} \right) \leq C \left( \|p\|_{L^\infty} + \|S\|_{L^\infty L^\infty} + \|S\|_{L^2 L^2} + \|z\|_{L^2 L^2} \right). \end{aligned}$$

Since  $\|p\|_{L^\infty}$ ,  $\|S\|_{L^2 L^2}$  and  $\|S\|_{L^\infty L^\infty}$  are finite, our result easily follows.  $\square$

Lemma 3.10 naturally implies the following.

**Theorem 3.11.** *The solution  $f(\eta, \phi)$  to the Milne problem (3.1) satisfies*

$$(3.216) \quad \|f - f_\infty\|_{L^\infty L^\infty} \leq C \left( 1 + M + \frac{M}{K} + \|f - f_\infty\|_{L^2 L^2} \right).$$

Combining Theorem 3.11 and Theorem 3.6, we deduce the main theorem.

**Theorem 3.12.** *There exists a unique solution  $f(\eta, \phi)$  to the  $\epsilon$ -Milne problem (3.1) satisfying*

$$(3.217) \quad \|f - f_\infty\|_{L^\infty L^\infty} \leq C \left( 1 + M + \frac{M}{K} \right).$$

**3.3. Exponential Decay.** In this section, we prove the spatial decay of the solution to the Milne problem.

**Theorem 3.13.** *Assume (3.3) and (3.4) hold. For  $K_0 > 0$  sufficiently small, the solution  $f(\eta, \phi)$  to the  $\epsilon$ -Milne problem (3.1) satisfies*

$$(3.218) \quad \|e^{K_0 \eta} (f - f_\infty)\|_{L^\infty L^\infty} \leq C \left( 1 + M + \frac{M}{K} \right),$$

*Proof.* Define  $Z = e^{K_0 \eta} g$  for  $z = f - f_\infty$ . We divide the analysis into several steps:

Step 1: We have

$$(3.219) \quad \|Z\|_{L^2 L^2}^2 = \int_0^\infty e^{2K_0 \eta} \left( \int_{-\pi}^\pi (f(\eta, \phi) - f_\infty)^2 d\phi \right) d\eta \leq C \left( 1 + M + \frac{M}{K} \right)^2.$$

*The proof of (3.219):* The orthogonal property (3.39) reveals

$$(3.220) \quad \langle f, f \sin \phi \rangle_\phi(\eta) = \langle r, r \sin \phi \rangle_\phi(\eta).$$

Multiplying  $e^{2K_0 \eta} f$  on both sides of equation (3.1) and integrating over  $\phi \in [-\pi, \pi]$ , we obtain

$$(3.221) \quad \begin{aligned} \frac{1}{2} \frac{d}{d\eta} \left( e^{2K_0 \eta} \langle r, r \sin \phi \rangle_\phi(\eta) \right) + \frac{1}{2} F(\eta) \left( e^{2K_0 \eta} \langle r, r \sin \phi \rangle_\phi(\eta) \right) - e^{2K_0 \eta} \left( K_0 \langle r, r \sin \phi \rangle_\phi(\eta) - \langle r, r \rangle_\phi(\eta) \right) \\ = e^{2K_0 \eta} \langle S, f \rangle_\phi(\eta). \end{aligned}$$

For  $K_0 < \min\{1/2, K\}$ , we have

$$(3.222) \quad \frac{3}{2} \|r(\eta)\|_{L^2}^2 \geq -K_0 \langle r, r \sin \phi \rangle_\phi(\eta) + \langle r, r \rangle_\phi(\eta) \geq \frac{1}{2} \|r(\eta)\|_{L^2}^2.$$

Similar to the proof of Lemma 3.3 and Lemma 3.4, formula as (3.221) and (3.222) imply

$$(3.223) \quad \|e^{K_0\eta}r\|_{L^2L^2}^2 = \int_0^\infty e^{2K_0\eta} \langle r, r \rangle_\phi(\eta) d\eta \leq C \left(1 + M + \frac{M}{K}\right)^2.$$

From (3.106), Cauchy's inequality and (3.9), we can deduce

$$(3.224) \quad \begin{aligned} & \int_0^\infty e^{2K_0\eta} \left( \int_{-\pi}^\pi (f(\eta, \phi) - f_\infty)^2 d\phi \right) d\eta \\ & \leq \int_0^\infty e^{2K_0\eta} \left( \int_{-\pi}^\pi r^2(\eta, \phi) d\phi \right) d\eta + \int_0^\infty e^{2K_0\eta} \left( \int_{-\pi}^\pi (q(\eta) - q_\infty)^2 d\phi \right) d\eta \\ & \leq \int_0^\infty e^{2K_0\eta} \|r(\eta)\|_{L^2}^2 d\eta \\ & \quad + \int_0^\infty e^{2K_0\eta} \left( \int_\eta^\infty |F(y)| \|r(y)\|_{L^2} dy \right)^2 d\eta + \int_0^\infty e^{2K_0\eta} \left( \int_\eta^\infty \|S(y)\|_{L^\infty} dy \right)^2 d\eta \\ & \leq C \left(1 + M + \frac{M}{K}\right)^2 \\ & \quad + C \left( \int_0^\infty e^{2K_0\eta} \|r(\eta)\|_{L^2}^2 d\eta \right) \left( \int_0^\infty \int_\eta^\infty e^{2K_0(\eta-y)} F^2(y) dy d\eta \right) + \int_0^\infty e^{2K_0\eta} \left( \int_\eta^\infty \|S(y)\|_{L^\infty} dy \right)^2 d\eta \\ & \leq C \left(1 + M + \frac{M}{K}\right)^2 \\ & \quad + C \left( \int_0^\infty e^{2K_0\eta} \|r(\eta)\|_{L^2}^2 d\eta \right) \left( \int_0^\infty \int_\eta^\infty F^2(y) dy d\eta \right) + \int_0^\infty e^{2K_0\eta} \left( \int_\eta^\infty \|S(y)\|_{L^\infty} dy \right)^2 d\eta \\ & \leq C \left(1 + M + \frac{M}{K}\right)^2. \end{aligned}$$

This completes the proof of (3.219) when  $\bar{S} = 0$ . By the method introduced in Lemma 3.5, we can extend above  $L^2$  estimates to the general  $S$  case. Note all the auxiliary functions constructed in Lemma 3.5 satisfy the estimates (3.219).

Step 2: We have

$$(3.225) \quad \|Z\|_{L^\infty L^\infty} \leq C \left(1 + M + \frac{M}{K} + \|Z\|_{L^2 L^2}\right).$$

*Proof of (3.225):*  $Z$  satisfies the equation

$$(3.226) \quad \begin{cases} \sin \phi \frac{\partial Z}{\partial \eta} + F(\eta) \cos \phi \frac{\partial Z}{\partial \phi} + Z &= \bar{Z} + e^{K_0\eta} S + K_0 \sin \phi Z, \\ Z(0, \phi) &= p(\phi) = h(\phi) - f_\infty \text{ for } \sin \phi > 0. \end{cases}$$

Since we know  $Z = \mathcal{A}p + \mathcal{T}(\bar{Z} + e^{K_0\eta} S + K_0 \sin \phi Z)$  leads to

$$(3.227) \quad \mathcal{T}(\bar{Z} + e^{K_0\eta} S + K_0 \sin \phi Z) = Z - \mathcal{A}p,$$

then by Lemma 3.9, (3.207) and (3.208), we can show

$$(3.228) \quad \begin{aligned} & \|Z - \mathcal{A}p\|_{L^\infty L^2} \\ & \leq C(\delta) \left( \|\bar{Z}\|_{L^2 L^2} + \|e^{K_0\eta} S\|_{L^2 L^2} + K_0 \|Z\|_{L^2 L^2} \right) + \delta \left( \|\bar{Z}\|_{L^\infty L^\infty} + \|e^{K_0\eta} S\|_{L^\infty L^\infty} + K_0 \|Z\|_{L^\infty L^\infty} \right) \\ & \leq C(\delta) \left( \|Z\|_{L^2 L^2} + \|e^{K_0\eta} S\|_{L^2 L^2} + K_0 \|Z\|_{L^2 L^2} \right) + \delta \left( \|Z\|_{L^\infty L^2} + \|e^{K_0\eta} S\|_{L^\infty L^\infty} + K_0 \|Z\|_{L^\infty L^\infty} \right). \end{aligned}$$

Therefore, based on Lemma 3.7 and (3.212), we can directly estimate

$$\begin{aligned}
 (3.229) \quad & \|Z\|_{L^\infty L^2} \\
 & \leq \| \mathcal{A}p \|_{L^\infty} + C(\delta) \left( \|Z\|_{L^2 L^2} + \|e^{K_0 \eta} S\|_{L^2 L^2} + K_0 \|Z\|_{L^2 L^2} \right) \\
 & \quad + \delta \left( \|Z\|_{L^\infty L^2} + \|e^{K_0 \eta} S\|_{L^\infty L^\infty} + K_0 \|Z\|_{L^\infty L^\infty} \right) \\
 & \leq \|p\|_{L^\infty} + 2C(\delta) \left( \|Z\|_{L^2 L^2} + \|e^{K_0 \eta} S\|_{L^2 L^2} \right) + \delta \left( \|Z\|_{L^\infty L^2} + \|e^{K_0 \eta} S\|_{L^\infty L^\infty} + K_0 \|Z\|_{L^\infty L^\infty} \right).
 \end{aligned}$$

We can take  $\delta = 1/2$  to obtain

$$(3.230) \quad \|Z\|_{L^\infty L^2} \leq C \left( \|Z\|_{L^2 L^2} + \|S\|_{L^2 L^2} + \|S\|_{L^\infty L^\infty} + \|p\|_{L^\infty} + K_0 \|Z\|_{L^\infty L^\infty} \right).$$

Then based on Lemma 3.7, Lemma 3.8 and Lemma 3.9, we can deduce

$$\begin{aligned}
 (3.231) \quad \|Z\|_{L^\infty L^\infty} & \leq \|e^{K_0 \eta} \mathcal{A}p\|_{L^\infty} + \|e^{K_0 \eta} \mathcal{T}S\|_{L^\infty L^\infty} + \|\bar{Z}\|_{L^\infty L^\infty} \\
 & \leq \|p\|_{L^\infty} + \|e^{K_0 \eta} S\|_{L^\infty L^\infty} + \|\bar{Z}\|_{L^\infty L^\infty} \\
 & \leq \|p\|_{L^\infty} + \|e^{K_0 \eta} S\|_{L^\infty L^\infty} + \|Z\|_{L^\infty L^2} \\
 & \leq C \left( \|Z\|_{L^2 L^2} + \|e^{K_0 \eta} S\|_{L^2 L^2} + \|e^{K_0 \eta} S\|_{L^\infty L^\infty} + \|p\|_{L^\infty} + K_0 \|Z\|_{L^\infty L^\infty} \right).
 \end{aligned}$$

Taking  $K_0$  sufficiently small, this completes the proof of (3.225).

Combining (3.219) and (3.225), we deduce (3.218).  $\square$

### 3.4. Maximum Principle.

**Theorem 3.14.** *The solution  $f(\eta, \phi)$  to the  $\epsilon$ -Milne problem (3.1) with  $S = 0$  satisfies the maximum principle, i.e.*

$$(3.232) \quad \min_{\sin \phi > 0} h(\phi) \leq f(\eta, \phi) \leq \max_{\sin \phi > 0} h(\phi).$$

*Proof.* We claim it is sufficient to show  $f(\eta, \phi) \leq 0$  whenever  $h(\phi) \leq 0$ . Suppose this claim is justified. Denote  $m = \min_{\sin \phi > 0} h(\phi)$  and  $M = \max_{\sin \phi > 0} h(\phi)$ . Then  $f^1 = f - M$  satisfies the equation

$$(3.233) \quad \begin{cases} \sin \phi \frac{\partial f^1}{\partial \eta} + F(\eta) \cos \phi \frac{\partial f^1}{\partial \phi} + f^1 - \bar{f}^1 & = 0, \\ f^1(0, \phi) & = h(\phi) - M \text{ for } \sin \phi > 0, \\ \lim_{\eta \rightarrow \infty} f^1(\eta, \phi) & = f_\infty^1. \end{cases}$$

Hence,  $h - M \leq 0$  implies  $f^1 \leq 0$  which is actually  $f \leq M$ . On the other hand,  $f^2 = m - f$  satisfies the equation

$$(3.234) \quad \begin{cases} \sin \phi \frac{\partial f^2}{\partial \eta} + F(\eta) \cos \phi \frac{\partial f^2}{\partial \phi} + f^2 - \bar{f}^2 & = 0, \\ f^2(0, \phi) & = m - h(\phi) \text{ for } \sin \phi > 0, \\ \lim_{\eta \rightarrow \infty} f^2(\eta, \phi) & = f_\infty^2. \end{cases}$$

Thus,  $m - h \leq 0$  implies  $f^2 \leq 0$  which further leads to  $f \geq m$ . Therefore, the maximum principle is established.

We now prove if  $h(\phi) \leq 0$ , we have  $f(\eta, \phi) \leq 0$ . We divide the proof into several steps:

Step 1: Penalized  $\epsilon$ -Milne problem in a finite slab.

Assuming  $h(\phi) \leq 0$ , we then consider the penalized Milne problem for  $f_\lambda^L(\eta, \phi)$  in the finite slab  $(\eta, \phi) \in [0, L] \times [-\pi, \pi)$

$$(3.235) \quad \begin{cases} \lambda f_\lambda^L + \sin \phi \frac{\partial f_\lambda^L}{\partial \eta} + F(\eta) \cos \phi \frac{\partial f_\lambda^L}{\partial \phi} + f_\lambda^L - \bar{f}_\lambda^L &= 0, \\ f_\lambda^L(0, \phi) &= h(\phi) \text{ for } \sin \phi < 0, \\ f_\lambda^L(L, \phi) &= f_\lambda^L(L, R\phi). \end{cases}$$

In order to construct the solution of (3.235), we iteratively define the sequence  $\{f_m^L\}_{m=1}^\infty$  as  $f_0^L = 0$  and

$$(3.236) \quad \begin{cases} \lambda f_m^L + \sin \phi \frac{\partial f_m^L}{\partial \eta} + F(\eta) \cos \phi \frac{\partial f_m^L}{\partial \phi} + f_m^L - \bar{f}_{m-1}^L &= 0, \\ f_m^L(0, \phi) &= h(\phi) \text{ for } \sin \phi < 0, \\ f_m^L(L, \phi) &= f_m^L(L, R\phi). \end{cases}$$

Along the characteristics, it is easy to see we always have  $f_m^L < 0$ . In the proof of Lemma 3.3, we have shown  $f_m^L$  converges strongly in  $L^\infty([0, L] \times [-\pi, \pi))$  to  $f_\lambda^L$  which satisfies (3.235). Also,  $f_\lambda^L$  satisfies

$$(3.237) \quad \|f_\lambda^L\|_{L^\infty L^\infty} \leq \frac{1+\lambda}{\lambda} \|h\|_{L^\infty}.$$

Naturally, we obtain  $f_\lambda^L \in L^2([0, L] \times [-\pi, \pi))$  and  $f_\lambda^L \leq 0$ .

Step 2:  $\epsilon$ -Milne problem in a finite slab.

Consider the Milne problem for  $f^L(\eta, \phi)$  in a finite slab  $(\eta, \phi) \in [0, L] \times [-\pi, \pi)$

$$(3.238) \quad \begin{cases} \sin \phi \frac{\partial f^L}{\partial \eta} + F(\eta) \cos \phi \frac{\partial f^L}{\partial \phi} + f^L - \bar{f}^L &= 0, \\ f^L(0, \phi) &= h(\phi) \text{ for } \sin \phi < 0, \\ f^L(L, \phi) &= f^L(L, R\phi). \end{cases}$$

In the proof of Lemma 3.3, we have shown  $f_\lambda^L$  is uniformly bounded in  $L^2([0, L] \times [-\pi, \pi))$  with respect to  $\lambda$ , which implies we can take weakly convergent subsequence  $f_\lambda^L \rightharpoonup f^L$  as  $\lambda \rightarrow 0$  with  $f^L \in L^2([0, L] \times [-\pi, \pi))$ . Naturally, we have  $f^L(\eta, \phi) \leq 0$ .

Step 3:  $\epsilon$ -Milne problem in an infinite slab.

Finally, in the proof of Lemma 3.4, by taking  $L \rightarrow \infty$ , we have

$$(3.239) \quad f^L \rightharpoonup f \text{ in } L_{loc}^2([0, L] \times [-\pi, \pi)),$$

where  $f$  satisfies (3.1). Certainly, we have  $f(\eta, \phi) \leq 0$ . This justifies the claim in Step 1. Hence, we complete the proof.  $\square$

**Remark 3.15.** Note that when  $F = 0$ , then all the previous proofs can be recovered and Theorem 3.12, Theorem 3.13 and Theorem 3.14 still hold. Hence, we can deduce the well-posedness, decay and maximum principle of the classical Milne problem

$$(3.240) \quad \begin{cases} \sin \phi \frac{\partial f}{\partial \eta} + f - \bar{f} &= S(\eta, \phi), \\ f(0, \phi) &= h(\phi) \text{ for } \sin \phi > 0, \\ \lim_{\eta \rightarrow \infty} f(\eta, \phi) &= f_\infty. \end{cases}$$

#### 4. PROOF OF THEOREM 1.2

We divide the proof into several steps:

Step 1: Remainder definitions.

We may rewrite the asymptotic expansion as follows:

$$(4.1) \quad u^\epsilon \sim \sum_{k=0}^{\infty} \epsilon^k U_k^\epsilon + \sum_{k=0}^{\infty} \epsilon^k \mathcal{W}_k^\epsilon.$$

The remainder can be defined as

$$(4.2) \quad R_N = u^\epsilon - \sum_{k=0}^N \epsilon^k U_k^\epsilon - \sum_{k=0}^N \epsilon^k \mathcal{U}_k^\epsilon = u^\epsilon - Q_N - \mathcal{Q}_N,$$

where

$$(4.3) \quad Q_N = \sum_{k=0}^N \epsilon^k U_k^\epsilon,$$

$$(4.4) \quad \mathcal{Q}_N = \sum_{k=0}^N \epsilon^k \mathcal{U}_k^\epsilon.$$

Noting the equation (1.41) is equivalent to the equation (1), we write  $\mathcal{L}$  to denote the neutron transport operator as follows:

$$(4.5) \quad \begin{aligned} \mathcal{L}u &= \epsilon \vec{w} \cdot \nabla_x u + u - \bar{u} \\ &= \sin \phi \frac{\partial u}{\partial \eta} - \frac{\epsilon}{1 - \epsilon \eta} \cos \phi \left( \frac{\partial u}{\partial \phi} + \frac{\partial u}{\partial \theta} \right) + u - \bar{u}. \end{aligned}$$

Step 2: Estimates of  $\mathcal{L}Q_N$ .

The interior contribution can be estimated as

$$(4.6) \quad \mathcal{L}Q_0 = \epsilon \vec{w} \cdot \nabla_x Q_0 + Q_0 - \bar{Q}_0 = \epsilon \vec{w} \cdot \nabla_x U_0^\epsilon + (U_0^\epsilon - \bar{U}_0^\epsilon) = \epsilon \vec{w} \cdot \nabla_x U_0^\epsilon.$$

We have

$$(4.7) \quad |\epsilon \vec{w} \cdot \nabla_x U_0^\epsilon| \leq C\epsilon |\nabla_x U_0^\epsilon| \leq C\epsilon.$$

This implies

$$(4.8) \quad |\mathcal{L}Q_0| \leq C\epsilon.$$

Similarly, for higher order term, we can estimate

$$(4.9) \quad \mathcal{L}Q_N = \epsilon \vec{w} \cdot \nabla_x Q_N + Q_N - \bar{Q}_N = \epsilon^{N+1} \vec{w} \cdot \nabla_x U_N^\epsilon.$$

We have

$$(4.10) \quad |\epsilon^{N+1} \vec{w} \cdot \nabla_x U_N^\epsilon| \leq C\epsilon^{N+1} |\nabla_x U_N^\epsilon| \leq C\epsilon^{N+1}.$$

This implies

$$(4.11) \quad |\mathcal{L}Q_N| \leq C\epsilon^{N+1}.$$

Step 3: Estimates of  $\mathcal{L}\mathcal{Q}_N$ .

The boundary layer solution is  $\mathcal{U}_k^\epsilon = (f_k^\epsilon - f_k^\epsilon(\infty)) \cdot \psi_0 = \mathcal{V}_k \psi_0$  where  $f_k^\epsilon(\eta, \theta, \phi)$  solves the  $\epsilon$ -Milne problem and  $\mathcal{V}_k = f_k^\epsilon - f_k^\epsilon(\infty)$ . Notice  $\psi_0 \psi = \psi_0$ , so the boundary layer contribution can be estimated as

$$(4.12) \quad \begin{aligned} \mathcal{L}\mathcal{Q}_0 &= \sin \phi \frac{\partial \mathcal{Q}_0}{\partial \eta} - \frac{\epsilon}{1 - \epsilon \eta} \cos \phi \left( \frac{\partial \mathcal{Q}_0}{\partial \phi} + \frac{\partial \mathcal{Q}_0}{\partial \theta} \right) + \mathcal{Q}_0 - \bar{\mathcal{Q}}_0 \\ &= \sin \phi \left( \psi_0 \frac{\partial \mathcal{V}_0}{\partial \eta} + \mathcal{V}_0 \frac{\partial \psi_0}{\partial \eta} \right) - \frac{\psi_0 \epsilon}{1 - \epsilon \eta} \cos \phi \left( \frac{\partial \mathcal{V}_0}{\partial \phi} + \frac{\partial \mathcal{V}_0}{\partial \theta} \right) + \psi_0 \mathcal{V}_0 - \psi_0 \bar{\mathcal{V}}_0 \\ &= \sin \phi \left( \psi_0 \frac{\partial \mathcal{V}_0}{\partial \eta} + \mathcal{V}_0 \frac{\partial \psi_0}{\partial \eta} \right) - \frac{\psi_0 \psi \epsilon}{1 - \epsilon \eta} \cos \phi \left( \frac{\partial \mathcal{V}_0}{\partial \phi} + \frac{\partial \mathcal{V}_0}{\partial \theta} \right) + \psi_0 \mathcal{V}_0 - \psi_0 \bar{\mathcal{V}}_0 \\ &= \psi_0 \left( \sin \phi \frac{\partial \mathcal{V}_0}{\partial \eta} - \frac{\epsilon \psi}{1 - \epsilon \eta} \cos \phi \frac{\partial \mathcal{V}_0}{\partial \phi} + \mathcal{V}_0 - \bar{\mathcal{V}}_0 \right) + \sin \phi \frac{\partial \psi_0}{\partial \eta} \mathcal{V}_0 - \frac{\psi_0 \epsilon}{1 - \epsilon \eta} \cos \phi \frac{\partial \mathcal{V}_0}{\partial \theta} \\ &= \sin \phi \frac{\partial \psi_0}{\partial \eta} \mathcal{V}_0 - \frac{\psi_0 \epsilon}{1 - \epsilon \eta} \cos \phi \frac{\partial \mathcal{V}_0}{\partial \theta}. \end{aligned}$$

Since  $\psi_0 = 1$  when  $\eta \leq 1/(4\epsilon)$ , the effective region of  $\partial_\eta \psi_0$  is  $\eta \geq 1/(4\epsilon)$  which is further and further from the origin as  $\epsilon \rightarrow 0$ . By Theorem 3.13, the first term in (4.12) can be controlled as

$$(4.13) \quad \left| \sin \phi \frac{\partial \psi_0}{\partial \eta} \gamma_0 \right| \leq C e^{-\frac{\kappa_0}{\epsilon}} \leq C\epsilon.$$

For the second term in (4.12), we have

$$(4.14) \quad \left| -\frac{\psi_0 \epsilon}{1 - \epsilon \eta} \cos \phi \frac{\partial \gamma_0}{\partial \theta} \right| \leq C\epsilon \left| \frac{\partial \gamma_0}{\partial \theta} \right| \leq C\epsilon.$$

This implies

$$(4.15) \quad |\mathcal{L}\mathcal{Q}_0| \leq C\epsilon.$$

Similarly, for higher order term, we can estimate

$$(4.16) \quad \begin{aligned} \mathcal{L}\mathcal{Q}_N &= \sin \phi \frac{\partial \mathcal{Q}_N}{\partial \eta} - \frac{\epsilon}{1 - \epsilon \eta} \cos \phi \left( \frac{\partial \mathcal{Q}_N}{\partial \phi} + \frac{\partial \mathcal{Q}_N}{\partial \theta} \right) + \mathcal{Q}_N - \bar{\mathcal{Q}}_N \\ &= \sum_{i=0}^k \epsilon^i \sin \phi \frac{\partial \psi_0}{\partial \eta} \gamma_i - \frac{\psi_0 \epsilon^{k+1}}{1 - \epsilon \eta} \cos \phi \frac{\partial \gamma_k}{\partial \theta}. \end{aligned}$$

Away from the origin, the first term in (4.16) can be controlled as

$$(4.17) \quad \left| \sum_{i=0}^k \epsilon^i \sin \phi \frac{\partial \psi_0}{\partial \eta} \gamma_i \right| \leq C e^{-\frac{\kappa_0}{\epsilon}} \leq C\epsilon^{k+1}.$$

For the second term in (4.16), we have

$$(4.18) \quad \left| -\frac{\psi_0 \epsilon^{k+1}}{1 - \epsilon \eta} \cos \phi \frac{\partial \gamma_k}{\partial \theta} \right| \leq C\epsilon^{k+1} \left| \frac{\partial \gamma_k}{\partial \theta} \right| \leq C\epsilon^{k+1}.$$

This implies

$$(4.19) \quad |\mathcal{L}\mathcal{Q}_N| \leq C\epsilon^{k+1}.$$

Step 4: Proof of (1.62).

In summary, since  $\mathcal{L}u^\epsilon = 0$ , collecting (4.2), (4.11) and (4.19), we can prove

$$(4.20) \quad |\mathcal{L}R_N| \leq C\epsilon^{N+1}.$$

Consider the asymptotic expansion to  $N = 3$ , then the remainder  $R_3$  satisfies the equation

$$(4.21) \quad \begin{cases} \epsilon \vec{w} \cdot \nabla_x R_3 + R_3 - \bar{R}_3 &= \mathcal{L}R_3 \text{ for } \vec{x} \in \Omega, \\ R_3(\vec{x}_0, \vec{w}) &= 0 \text{ for } \vec{w} \cdot \vec{n} < 0 \text{ and } \vec{x}_0 \in \partial\Omega. \end{cases}$$

By Theorem 2.6, we have

$$(4.22) \quad \|R_3\|_{L^\infty(\Omega \times \mathcal{S}^1)} \leq \frac{C(\Omega)}{\epsilon^3} \|\mathcal{L}R_3\|_{L^\infty(\Omega \times \mathcal{S}^1)} \leq \frac{C(\Omega)}{\epsilon^3} (C\epsilon^4) = C(\Omega)\epsilon.$$

Hence, we have

$$(4.23) \quad \left\| u^\epsilon - \sum_{k=0}^3 \epsilon^k U_k^\epsilon - \sum_{k=0}^3 \epsilon^k \mathcal{W}_k^\epsilon \right\|_{L^\infty(\Omega \times \mathcal{S}^1)} = O(\epsilon).$$

Since it is easy to see

$$(4.24) \quad \left\| \sum_{k=1}^3 \epsilon^k U_k^\epsilon + \sum_{k=1}^3 \epsilon^k \mathcal{W}_k^\epsilon \right\|_{L^\infty(\Omega \times \mathcal{S}^1)} = O(\epsilon),$$

our result naturally follows. This completes the proof of (1.62).



Step 5: Basic settings to show (1.63).

By (1.28), the solution  $f_0$  satisfies the Milne problem

$$(4.25) \quad \begin{cases} \sin(\theta + \xi) \frac{\partial f_0}{\partial \eta} + f_0 - \bar{f}_0 &= 0, \\ f_0(0, \theta, \xi) &= g(\theta, \xi) \text{ for } \sin(\theta + \xi) > 0, \\ \lim_{\eta \rightarrow \infty} f_0(\eta, \theta, \xi) &= f_0(\infty, \theta). \end{cases}$$

For convenience of comparison, we make the substitution  $\phi = \theta + \xi$  to obtain

$$(4.26) \quad \begin{cases} \sin \phi \frac{\partial f_0}{\partial \eta} + f_0 - \bar{f}_0 &= 0, \\ f_0(0, \theta, \phi) &= g(\theta, \phi) \text{ for } \sin \phi > 0, \\ \lim_{\eta \rightarrow \infty} f_0(\eta, \theta, \phi) &= f_0(\infty, \theta). \end{cases}$$

Assume (1.63) is incorrect, such that

$$(4.27) \quad \lim_{\epsilon \rightarrow 0} \|(U_0 + \mathcal{U}_0) - (U_0^\epsilon + \mathcal{U}_0^\epsilon)\|_{L^\infty} = 0.$$

Since the boundary  $g(\phi) = \cos \phi$  independent of  $\theta$ , by (1.28) and (1.53), it is obvious the limit of zeroth order boundary layer  $f_0(\infty, \theta)$  and  $f_0^\epsilon(\infty, \theta)$  satisfy  $f_0(\infty, \theta) = C_1$  and  $f_0^\epsilon(\infty, \theta) = C_2$  for some constant  $C_1$  and  $C_2$  independent of  $\theta$ . By (1.29) and (1.54), we can derive the interior solutions are indeed constants  $U_0 = C_1$  and  $U_0^\epsilon = C_2$ . Hence, we may further derive

$$(4.28) \quad \lim_{\epsilon \rightarrow 0} \|(f_0(\infty) + \mathcal{U}_0) - (f_0^\epsilon(\infty) + \mathcal{U}_0^\epsilon)\|_{L^\infty} = 0.$$

For  $0 \leq \eta \leq 1/(2\epsilon)$ , we have  $\psi_0 = 1$ , which means  $f_0 = \mathcal{U}_0 + f_0(\infty)$  and  $f_0^\epsilon = \mathcal{U}_0^\epsilon + f_0^\epsilon(\infty)$  on  $[0, 1/(2\epsilon)]$ . Define  $u = f_0 + 2$ ,  $U = f_0^\epsilon + 2$  and  $G = g + 2 = \cos \phi + 2$ , then  $u(\eta, \phi)$  satisfies the equation

$$(4.29) \quad \begin{cases} \sin \phi \frac{\partial u}{\partial \eta} + u - \bar{u} &= 0, \\ u(0, \phi) &= G(\phi) \text{ for } \sin \phi > 0, \\ \lim_{\eta \rightarrow \infty} u(\eta, \phi) &= 2 + f_0(\infty), \end{cases}$$

and  $U(\eta, \phi)$  satisfies the equation

$$(4.30) \quad \begin{cases} \sin \phi \frac{\partial U}{\partial \eta} + F(\epsilon; \eta) \cos \phi \frac{\partial U}{\partial \phi} + U - \bar{U} &= 0, \\ U(0, \phi) &= G(\phi) \text{ for } \sin \phi > 0, \\ \lim_{\eta \rightarrow \infty} U(\eta, \phi) &= 2 + f_0^\epsilon(\infty). \end{cases}$$

Based on (4.28), we have

$$(4.31) \quad \lim_{\epsilon \rightarrow 0} \|U(\eta, \phi) - u(\eta, \phi)\|_{L^\infty} = 0.$$

Then it naturally implies

$$(4.32) \quad \lim_{\epsilon \rightarrow 0} \|\bar{U}(\eta) - \bar{u}(\eta)\|_{L^\infty} = 0.$$

Step 6: Continuity of  $\bar{u}$  and  $\bar{U}$  at  $\eta = 0$ .

For the problem (4.29), we have for any  $r_0 > 0$

$$(4.33) \quad |\bar{u}(\eta) - \bar{u}(0)| \leq \frac{1}{2\pi} \left( \int_{\sin \phi \leq r_0} |u(\eta, \phi) - u(0, \phi)| d\phi + \int_{\sin \phi \geq r_0} |u(\eta, \phi) - u(0, \phi)| d\phi \right).$$

Since we have shown  $u \in L^\infty([0, \infty) \times [-\pi, \pi])$ , then for any  $\delta > 0$ , we can take  $r_0$  sufficiently small such that

$$(4.34) \quad \frac{1}{2\pi} \int_{\sin \phi \leq r_0} |u(\eta, \phi) - u(0, \phi)| d\phi \leq \frac{C}{2\pi} \arcsin r_0 \leq \frac{\delta}{2}.$$

For fixed  $r_0$  satisfying above requirement, we estimate the integral on  $\sin \phi \geq r_0$ . By Ukai's trace theorem,  $u(0, \phi)$  is well-defined in the domain  $\sin \phi \geq r_0$  and is continuous. Also, by consider the relation

$$(4.35) \quad \frac{\partial u}{\partial \eta}(0, \phi) = \frac{\bar{u}(0) - u(0, \phi)}{\sin \phi},$$

we can obtain in this domain  $\partial_\eta u$  is bounded, which further implies  $u(\eta, \phi)$  is uniformly continuous at  $\eta = 0$ . Then there exists  $\delta_0 > 0$  sufficiently small, such that for any  $0 \leq \eta \leq \delta_0$ , we have

$$(4.36) \quad \frac{1}{2\pi} \int_{\sin \phi \geq r_0} |u(\eta, \phi) - u(0, \phi)| d\phi \leq \frac{1}{2\pi} \int_{\sin \phi \geq r_0} \frac{\delta}{2} d\phi \leq \frac{\delta}{2}.$$

In summary, we have shown for any  $\delta > 0$ , there exists  $\delta_0 > 0$  such that for any  $0 \leq \eta \leq \delta_0$ ,

$$(4.37) \quad |\bar{u}(\eta) - \bar{u}(0)| \leq \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

Hence,  $\bar{u}(\eta)$  is continuous at  $\eta = 0$ . By a similar argument along the characteristics, we can show  $\bar{U}(\eta, \phi)$  is also continuous at  $\eta = 0$ .

In the following, by the continuity, we assume for arbitrary  $\delta > 0$ , there exists a  $\delta_0 > 0$  such that for any  $0 \leq \eta \leq \delta_0$ , we have

$$(4.38) \quad |\bar{u}(\eta) - \bar{u}(0)| \leq \delta,$$

$$(4.39) \quad |\bar{U}(\eta) - \bar{U}(0)| \leq \delta.$$

Step 7 Milne formulation.

We consider the solution at a specific point  $(\eta, \phi) = (n\epsilon, \epsilon)$  for some fixed  $n > 0$ . The solution along the characteristics can be rewritten as follows:

$$(4.40) \quad u(n\epsilon, \epsilon) = G(\epsilon) e^{-\frac{1}{\sin \epsilon} n\epsilon} + \int_0^{n\epsilon} e^{-\frac{1}{\sin \epsilon} (n\epsilon - \kappa)} \frac{1}{\sin \epsilon} \bar{u}(\kappa) d\kappa,$$

$$(4.41) \quad U(n\epsilon, \epsilon) = G(\epsilon_0) e^{-\int_0^{n\epsilon} \frac{1}{\sin \phi(\zeta)} d\zeta} + \int_0^{n\epsilon} e^{-\int_\kappa^{n\epsilon} \frac{1}{\sin \phi(\zeta)} d\zeta} \frac{1}{\sin \phi(\kappa)} \bar{U}(\kappa) d\kappa,$$

where we have the conserved energy along the characteristics

$$(4.42) \quad E(\eta, \phi) = \cos \phi e^{-V(\eta)},$$

in which  $(0, \epsilon_0)$  and  $(\zeta, \phi(\zeta))$  are in the same characteristics of  $(n\epsilon, \epsilon)$ .

Step 8: Estimates of (4.40).

We turn to the Milne problem for  $u$ . We have the natural estimate

$$(4.43) \quad \begin{aligned} \int_0^{n\epsilon} e^{-\frac{1}{\sin \epsilon} (n\epsilon - \kappa)} \frac{1}{\sin \epsilon} d\kappa &= \int_0^{n\epsilon} e^{-\frac{1}{\epsilon} (n\epsilon - \kappa)} \frac{1}{\epsilon} d\kappa + o(\epsilon) \\ &= e^{-n} \int_0^{n\epsilon} e^{\frac{\kappa}{\epsilon}} \frac{1}{\epsilon} d\kappa + o(\epsilon) \\ &= e^{-n} \int_0^n e^\zeta d\zeta + o(\epsilon) \\ &= (1 - e^{-n}) + o(\epsilon). \end{aligned}$$

Then for  $0 < \epsilon \leq \delta_0$ , we have  $|\bar{u}(0) - \bar{u}(\kappa)| \leq \delta$ , which implies

$$(4.44) \quad \begin{aligned} \int_0^{n\epsilon} e^{-\frac{1}{\sin \epsilon} (n\epsilon - \kappa)} \frac{1}{\sin \epsilon} \bar{u}(\kappa) d\kappa &= \int_0^{n\epsilon} e^{-\frac{1}{\sin \epsilon} (n\epsilon - \kappa)} \frac{1}{\sin \epsilon} \bar{u}(0) d\kappa + O(\delta) \\ &= (1 - e^{-n}) \bar{u}(0) + o(\epsilon) + O(\delta). \end{aligned}$$

For the boundary data term, it is easy to see

$$(4.45) \quad G(\epsilon) e^{-\frac{1}{\sin \epsilon} n\epsilon} = e^{-n} G(\epsilon) + o(\epsilon)$$

In summary, we have

$$(4.46) \quad u(n\epsilon, \epsilon) = (1 - e^{-n}) \bar{u}(0) + e^{-n} G(\epsilon) + o(\epsilon) + O(\delta).$$

Step 9: Estimates of (4.41).

We consider the  $\epsilon$ -Milne problem for  $U$ . For  $\epsilon < 1$  sufficiently small,  $\psi(\epsilon) = 1$ . Then we may estimate

$$(4.47) \quad \cos \phi(\zeta) e^{-V(\zeta)} = \cos \epsilon e^{-V(n\epsilon)},$$

which implies

$$(4.48) \quad \cos \phi(\zeta) = \frac{1 - n\epsilon^2}{1 - \epsilon\zeta} \cos \epsilon.$$

and hence

$$(4.49) \quad \sin \phi(\zeta) = \sqrt{1 - \cos^2 \phi(\zeta)} = \sqrt{\frac{\epsilon(n\epsilon - \zeta)(2 - \epsilon\zeta - n\epsilon^2)}{(1 - \epsilon\zeta)^2} \cos^2 \epsilon + \sin^2 \epsilon}.$$

For  $\zeta \in [0, \epsilon]$  and  $n\epsilon$  sufficiently small, by Taylor's expansion, we have

$$(4.50) \quad 1 - \epsilon\zeta = 1 + o(\epsilon),$$

$$(4.51) \quad 2 - \epsilon\zeta - n\epsilon^2 = 2 + o(\epsilon),$$

$$(4.52) \quad \sin^2 \epsilon = \epsilon^2 + o(\epsilon^3),$$

$$(4.53) \quad \cos^2 \epsilon = 1 - \epsilon^2 + o(\epsilon^3).$$

Hence, we have

$$(4.54) \quad \sin \phi(\zeta) = \sqrt{\epsilon(\epsilon + 2n\epsilon - 2\zeta)} + o(\epsilon^2).$$

Since  $\sqrt{\epsilon(\epsilon + 2n\epsilon - 2\zeta)} = O(\epsilon)$ , we can further estimate

$$(4.55) \quad \frac{1}{\sin \phi(\zeta)} = \frac{1}{\sqrt{\epsilon(\epsilon + 2n\epsilon - 2\zeta)}} + o(1)$$

$$(4.56) \quad - \int_{\kappa}^{n\epsilon} \frac{1}{\sin \phi(\zeta)} d\zeta = \sqrt{\frac{\epsilon + 2n\epsilon - 2\zeta}{\epsilon}} \Big|_{\kappa}^{n\epsilon} + o(\epsilon) = 1 - \sqrt{\frac{\epsilon + 2n\epsilon - 2\kappa}{\epsilon}} + o(\epsilon).$$

Then we can easily derive the integral estimate

$$(4.57) \quad \begin{aligned} \int_0^{n\epsilon} e^{-\int_{\kappa}^{n\epsilon} \frac{1}{\sin \phi(\zeta)} d\zeta} \frac{1}{\sin \phi(\kappa)} d\kappa &= e^1 \int_0^{n\epsilon} e^{-\sqrt{\frac{\epsilon + 2n\epsilon - 2\kappa}{\epsilon}}} \frac{1}{\sqrt{\epsilon(\epsilon + 2n\epsilon - 2\kappa)}} d\kappa + o(\epsilon) \\ &= \frac{1}{2} e^1 \int_{\epsilon}^{(1+2n)\epsilon} e^{-\sqrt{\frac{\sigma}{\epsilon}}} \frac{1}{\sqrt{\epsilon\sigma}} d\sigma + o(\epsilon) \\ &= \frac{1}{2} e^1 \int_1^{1+2n} e^{-\sqrt{\rho}} \frac{1}{\sqrt{\rho}} d\rho + o(\epsilon) \\ &= e^1 \int_1^{\sqrt{1+2n}} e^{-t} dt + o(\epsilon) \\ &= (1 - e^{1-\sqrt{1+2n}}) + o(\epsilon). \end{aligned}$$

Then for  $0 < \epsilon \leq \delta_0$ , we have  $|\bar{U}(0) - \bar{U}(\kappa)| \leq \delta$ , which implies

$$(4.58) \quad \begin{aligned} \int_0^{n\epsilon} e^{-\int_{\kappa}^{n\epsilon} \frac{1}{\sin \phi(\zeta)} d\zeta} \frac{1}{\sin \phi(\kappa)} \bar{U}(\kappa) d\kappa &= \int_0^{n\epsilon} e^{-\int_{\kappa}^{n\epsilon} \frac{1}{\sin \phi(\zeta)} d\zeta} \frac{1}{\sin \phi(\kappa)} \bar{U}(0) d\kappa + O(\delta) \\ &= (1 - e^{1-\sqrt{1+2n}}) \bar{U}(0) + o(\epsilon) + O(\delta). \end{aligned}$$

For the boundary data term, since  $G(\phi)$  is  $C^1$ , a similar argument shows

$$(4.59) \quad G(\epsilon_0) e^{-\int_0^{n\epsilon} \frac{1}{\sin \phi(\zeta)} d\zeta} = e^{1-\sqrt{1+2n}} G(\sqrt{1+2n}\epsilon) + o(\epsilon).$$

Therefore, we have

$$(4.60) \quad U(n\epsilon, \epsilon) = (1 - e^{1-\sqrt{1+2n}}) \bar{U}(0) + e^{1-\sqrt{1+2n}} G(\sqrt{1+2n}\epsilon) + o(\epsilon) + O(\delta).$$

Step 10: Proof of (1.63).

In summary, we have the estimate

$$(4.61) \quad u(n\epsilon, \epsilon) = (1 - e^{-n}) \bar{u}(0) + e^{-n} G(\epsilon) + o(\epsilon) + O(\delta),$$

$$(4.62) \quad U(n\epsilon, \epsilon) = (1 - e^{1-\sqrt{1+2n}}) \bar{U}(0) + e^{1-\sqrt{1+2n}} G(\sqrt{1+2n}\epsilon) + o(\epsilon) + O(\delta).$$

The boundary data is  $G = \cos \phi + 2$ . Then by the maximum principle in Theorem 3.14, we can achieve  $1 \leq u(0, \phi) \leq 3$  and  $1 \leq U(0, \phi) \leq 3$ . Since

$$\begin{aligned}
 (4.63) \quad \bar{u}(0) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} u(0, \phi) d\phi = \frac{1}{2\pi} \int_{\sin \phi > 0} u(0, \phi) d\phi + \frac{1}{2\pi} \int_{\sin \phi < 0} u(0, \phi) d\phi \\
 &= \frac{1}{2\pi} \int_{\sin \phi > 0} \cos \phi d\phi + \frac{1}{2\pi} \int_{\sin \phi < 0} u(0, \phi) d\phi \\
 &= \frac{1}{2\pi} \int_{\sin \phi < 0} M d\phi + \frac{1}{2\pi} \int_{\sin \phi < 0} u(0, \phi) d\phi,
 \end{aligned}$$

we naturally obtain  $3/2 \leq \bar{u}(0) \leq 5/2$ . Similarly, we can obtain  $3/2 \leq \bar{U}(0) \leq 5/2$ . Furthermore, for  $\epsilon$  sufficiently small, we have

$$(4.64) \quad G(\sqrt{1+2n\epsilon}) = 3 + o(\epsilon),$$

$$(4.65) \quad G(\epsilon) = 3 + o(\epsilon).$$

Hence, we can obtain

$$(4.66) \quad u(n\epsilon, \epsilon) = \bar{u}(0) + e^{-n}(-\bar{u}(0) + 3) + o(\epsilon) + O(\delta),$$

$$(4.67) \quad U(n\epsilon, \epsilon) = \bar{U}(0) + e^{1-\sqrt{1+2n}}(-\bar{U}(0) + 3) + o(\epsilon) + O(\delta).$$

Then we can see  $\lim_{\epsilon \rightarrow 0} \|\bar{U}(0) - \bar{u}(0)\|_{L^\infty} = 0$  naturally leads to  $\lim_{\epsilon \rightarrow 0} \|(-\bar{u}(0) + 3) - (-\bar{U}(0) + 3)\|_{L^\infty} = 0$ . Also, we have  $-\bar{u}(0) + 3 = O(1)$  and  $-\bar{U}(0) + 3 = O(1)$ . Due to the smallness of  $\epsilon$  and  $\delta$ , and also  $e^{-n} \neq e^{1-\sqrt{1+2n}}$ , we can obtain

$$(4.68) \quad |U(n\epsilon, \epsilon) - u(n\epsilon, \epsilon)| = O(1).$$

However, above result contradicts our assumption that  $\lim_{\epsilon \rightarrow 0} \|U(\eta, \phi) - u(\eta, \phi)\|_{L^\infty} = 0$  for any  $(\eta, \phi)$ . This completes the proof of (1.63).

## 5. NEUTRON TRANSPORT EQUATION WITH DIFFUSIVE BOUNDARY

**5.1. Problem Settings.** We consider the steady neutron transport equation in a two-dimensional unit disk with diffusive boundary. In the space domain  $\Omega = \{\vec{x} : |\vec{x}| \leq 1\}$  and the velocity domain  $\Sigma = \{\vec{w} : \vec{w} \in \mathcal{S}^1\}$ , the neutron density  $u^\epsilon(\vec{x}, \vec{w})$  satisfies

$$(5.1) \quad \begin{cases} \epsilon \vec{w} \cdot \nabla_x u^\epsilon + (1 + \epsilon^2)u^\epsilon - \bar{u}^\epsilon &= 0 \text{ in } \Omega, \\ u^\epsilon(\vec{x}_0, \vec{w}) &= \mathcal{P}u^\epsilon(\vec{x}_0) + \epsilon g(\vec{x}_0, \vec{w}) \text{ for } \vec{w} \cdot \vec{n} < 0 \text{ and } \vec{x}_0 \in \partial\Omega, \end{cases}$$

where

$$(5.2) \quad \bar{u}^\epsilon(\vec{x}) = \frac{1}{2\pi} \int_{\mathcal{S}^1} u^\epsilon(\vec{x}, \vec{w}) d\vec{w},$$

$$(5.3) \quad \mathcal{P}u^\epsilon(\vec{x}_0) = \frac{1}{2} \int_{\vec{w} \cdot \vec{n} > 0} u^\epsilon(\vec{x}_0, \vec{w}) (\vec{w} \cdot \vec{n}) d\vec{w},$$

with the Knudsen number  $0 < \epsilon \ll 1$ .

**5.2. Interior Expansion.** We define the interior expansion as follows:

$$(5.4) \quad U(\vec{x}, \vec{w}) \sim \sum_{k=0}^{\infty} \epsilon^k U_k(\vec{x}, \vec{w}),$$

where  $U_k$  can be defined by comparing the order of  $\epsilon$  by plugging (5.4) into the equation (5.1). Thus we have

$$(5.5) \quad U_0 - \bar{U}_0 = 0,$$

$$(5.6) \quad U_1 - \bar{U}_1 = -\vec{w} \cdot \nabla_x U_0,$$

$$(5.7) \quad U_2 - \bar{U}_2 = -\vec{w} \cdot \nabla_x U_1 - U_0,$$

$\dots$

$$(5.8) \quad U_k - \bar{U}_k = -\vec{w} \cdot \nabla_x U_{k-1} - U_{k-2}.$$

The following analysis reveals the equation satisfied by  $U_k$ :

Plugging (5.5) into (5.6), we obtain

$$(5.9) \quad U_1 = \bar{U}_1 - \vec{w} \cdot \nabla_x \bar{U}_0.$$

Plugging (5.9) into (5.7), we get

$$(5.10) \quad U_2 - \bar{U}_2 = -\vec{w} \cdot \nabla_x (\bar{U}_1 - \vec{w} \cdot \nabla_x \bar{U}_0) = -\vec{w} \cdot \nabla_x \bar{U}_1 + \vec{w}^2 \Delta_x \bar{U}_0 + 2w_x w_y \partial_{xy} \bar{U}_0 - U_0.$$

Integrating (5.10) over  $\vec{w} \in \mathcal{S}^1$ , we achieve the final form

$$(5.11) \quad \Delta_x \bar{U}_0 - \bar{U}_0 = 0.$$

which further implies  $U_0(\vec{x}, \vec{w})$  satisfies the equation

$$(5.12) \quad \begin{cases} U_0 &= \bar{U}_0, \\ \Delta_x \bar{U}_0 - \bar{U}_0 &= 0. \end{cases}$$

In a similar fashion,  $U_1$  satisfies

$$(5.13) \quad \begin{cases} U_1 &= \bar{U}_1 - \vec{w} \cdot \nabla_x U_0, \\ \Delta_x \bar{U}_1 - \bar{U}_1 &= - \int_{\mathcal{S}^1} \vec{w} \cdot \nabla_x U_0 d\vec{w}. \end{cases}$$

Also,  $U_k(\vec{x}, \vec{w})$  for  $k \geq 2$  satisfies

$$(5.14) \quad \begin{cases} U_k &= \bar{U}_k - \vec{w} \cdot \nabla_x U_{k-1} - U_{k-2}, \\ \Delta_x \bar{U}_k - \bar{U}_k &= - \int_{\mathcal{S}^1} \vec{w} \cdot \nabla_x U_{k-1} d\vec{w} - \int_{\mathcal{S}^1} U_{k-2} d\vec{w}. \end{cases}$$

**5.3. Milne Expansion.** In order to determine the boundary condition for  $U_k$ , it is well-known that we need to define the boundary layer expansion. We still perform the substitutions (1.14), (1.16) and (1.19) to the equation (5.1) to obtain the form

$$(5.15) \quad \begin{cases} \sin(\theta + \xi) \frac{\partial u^\epsilon}{\partial \eta} - \frac{\epsilon}{1 - \epsilon \eta} \cos(\theta + \xi) \frac{\partial u^\epsilon}{\partial \theta} + (1 + \epsilon^2) u^\epsilon - \frac{1}{2\pi} \int_{-\pi}^{\pi} u^\epsilon d\xi = 0, \\ u^\epsilon(0, \theta, \xi) = \mathcal{P}u^\epsilon(0, \theta) + \epsilon g(\theta, \xi) \text{ for } \sin(\theta + \xi) > 0, \end{cases}$$

where

$$(5.16) \quad \mathcal{P}u^\epsilon(0, \theta) = -\frac{1}{2} \int_{\sin(\theta + \xi) < 0} u^\epsilon(0, \theta, \xi) \sin(\theta + \xi) d\xi.$$

Define the Milne expansion of boundary layer as follows:

$$(5.17) \quad \mathcal{U}(\eta, \theta, \phi) \sim \sum_{k=0}^{\infty} \epsilon^k \mathcal{U}_k(\eta, \theta, \phi)$$

where  $\mathcal{U}_k$  is defined by comparing the order of  $\epsilon$  via plugging (5.17) into the equation (5.15). Thus, in a neighborhood of the boundary, we have

$$(5.18) \quad \sin(\theta + \xi) \frac{\partial \mathcal{U}_0}{\partial \eta} + \mathcal{U}_0 - \bar{\mathcal{U}}_0 = 0,$$

$$(5.19) \quad \sin(\theta + \xi) \frac{\partial \mathcal{U}_1}{\partial \eta} + \mathcal{U}_1 - \bar{\mathcal{U}}_1 = \frac{1}{1 - \epsilon \eta} \cos(\theta + \xi) \frac{\partial \mathcal{U}_0}{\partial \theta},$$

$$(5.20) \quad \sin(\theta + \xi) \frac{\partial \mathcal{U}_2}{\partial \eta} + \mathcal{U}_2 - \bar{\mathcal{U}}_2 = \frac{1}{1 - \epsilon \eta} \cos(\theta + \xi) \frac{\partial \mathcal{U}_1}{\partial \theta} - \mathcal{U}_0,$$

$$(5.21) \quad \begin{aligned} &\dots \\ \sin(\theta + \xi) \frac{\partial \mathcal{U}_k}{\partial \eta} + \mathcal{U}_k - \bar{\mathcal{U}}_k &= \frac{1}{1 - \epsilon \eta} \cos(\theta + \xi) \frac{\partial \mathcal{U}_{k-1}}{\partial \theta} - \mathcal{U}_{k-2}, \end{aligned}$$

where

$$(5.22) \quad \bar{\mathcal{U}}_k(\eta, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathcal{U}_k(\eta, \theta, \xi) d\xi.$$

The construction of  $U_k$  and  $\mathcal{U}_k$  in [4] can be summarized as follows:

Step 1: Construction of  $\mathcal{U}_0$ .

Define  $\psi$  and  $\psi_0$  as (1.26) and (1.27). Then the zeroth order boundary layer solution can be defined as

$$(5.23) \quad \begin{cases} \mathcal{U}_0 &= \psi_0(\epsilon\eta) \left( f_0(\eta, \theta, \xi) - f_0(\infty, \theta) \right), \\ \sin(\theta + \xi) \frac{\partial f_0}{\partial \eta} + f_0 - \bar{f}_0 &= 0, \\ f_0(0, \theta, \xi) &= \mathcal{P}f_0(0, \theta) \quad \text{for } \sin(\theta + \xi) > 0, \\ \lim_{\eta \rightarrow \infty} f_0(\eta, \theta, \xi) &= f_0(\infty, \theta), \end{cases}$$

with

$$(5.24) \quad \mathcal{P}f_0(0, \theta) = 0.$$

It is easy to see  $\mathcal{U}_0 = f_0 = 0$ .

Step 2: Construction of  $\mathcal{U}_1$  and  $U_0$ .

Define the first order boundary layer solution as

$$(5.25) \quad \begin{cases} \mathcal{U}_1 &= \psi_0(\epsilon\eta) \left( f_1(\eta, \theta, \xi) - f_1(\infty, \theta) \right), \\ \sin(\theta + \xi) \frac{\partial f_1}{\partial \eta} + f_1 - \bar{f}_1 &= \cos(\theta + \xi) \frac{\psi(\epsilon\eta)}{1 - \epsilon\eta} \frac{\partial \mathcal{U}_0}{\partial \theta}, \\ f_1(0, \theta, \xi) &= \mathcal{P}f_1(0, \theta) + g_1(\theta, \xi) \quad \text{for } \sin(\theta + \xi) > 0, \\ \lim_{\eta \rightarrow \infty} f_1(\eta, \theta, \xi) &= f_1(\infty, \theta), \end{cases}$$

with

$$(5.26) \quad \mathcal{P}f_1(0, \theta) = 0,$$

where  $g_1 = \vec{w} \cdot \nabla_x U_0 - \mathcal{P}(\vec{w} \cdot \nabla_x U_0) + g$  in which  $(\vec{x}_0, \vec{w})$  and  $(0, \theta, \xi)$  denote the same point. Since  $\mathcal{U}_0 = 0$ , we can obtain a unique  $f_1(\eta, \theta, \xi) \in L^\infty([0, \infty) \times [-\pi, \pi] \times [-\pi, \pi])$ . Hence,  $\mathcal{U}_1$  is well-defined. By the compatibility condition (5.99), we can define the zeroth order interior solution as

$$(5.27) \quad \begin{cases} U_0 &= \bar{U}_0, \\ \Delta_x \bar{U}_0 - \bar{U}_0 &= 0 \quad \text{in } \Omega, \\ \frac{\partial U_0}{\partial \vec{n}} &= \frac{1}{\pi} \int_{\sin(\theta+\xi)>0} g(\theta, \xi) \sin(\theta + \xi) d\xi \quad \text{on } \partial\Omega. \end{cases}$$

Step 3: Construction of  $\mathcal{U}_2$  and  $U_1$ .

Define the second order boundary layer solution as

$$(5.28) \quad \begin{cases} \mathcal{U}_2 &= \psi_0(\epsilon\eta) \left( f_2(\eta, \theta, \xi) - f_2(\infty, \theta) \right), \\ \sin(\theta + \xi) \frac{\partial f_2}{\partial \eta} + f_2 - \bar{f}_2 &= \cos(\theta + \xi) \frac{\psi(\epsilon\eta)}{1 - \epsilon\eta} \frac{\partial \mathcal{U}_1}{\partial \theta} - \psi(\epsilon\eta) \mathcal{U}_0(\eta, \theta, \xi), \\ f_2(0, \theta, \xi) &= \mathcal{P}f_2(0, \theta) + g_2(\theta, \xi) \quad \text{for } \sin(\theta + \xi) > 0, \\ \lim_{\eta \rightarrow \infty} f_2(\eta, \theta, \xi) &= f_2(\infty, \theta), \end{cases}$$

with

$$(5.29) \quad \mathcal{P}f_2(0, \theta) = 0.$$

where  $g_2 = \vec{w} \cdot \nabla_x U_1 - \mathcal{P}(\vec{w} \cdot \nabla_x U_1) + U_0 - \mathcal{P}U_0$ . By the compatibility condition (5.99), we define the first order interior solution as

$$(5.30) \quad \begin{cases} U_1 &= \bar{U}_1 - \vec{w} \cdot \nabla_x U_0, \\ \Delta_x \bar{U}_1 - \bar{U}_1 &= - \int_{S^1} (\vec{w} \cdot \nabla_x U_0) d\vec{w} \quad \text{in } \Omega, \\ \frac{\partial U_1}{\partial \vec{n}} &= \frac{1}{\pi} \int_0^\infty \int_{-\pi}^\pi \frac{\psi(\epsilon s)}{1 - \epsilon s} \cos(\theta + \xi) \frac{\partial \mathcal{U}_1}{\partial \theta}(s, \theta, \xi) d\xi ds \quad \text{on } \partial\Omega. \end{cases}$$

Step 4: Generalization to arbitrary  $k$ .

Define the  $k^{th}$  order boundary layer solution as

$$(5.31) \quad \begin{cases} \mathcal{U}_k &= \psi_0(\epsilon\eta) \left( f_k(\eta, \theta, \xi) - f_k(\infty, \theta) \right), \\ \sin(\theta + \xi) \frac{\partial f_k}{\partial \eta} + f_k - \bar{f}_k &= \cos(\theta + \xi) \frac{\psi(\epsilon\eta)}{1 - \epsilon\eta} \frac{\partial \mathcal{U}_{k-1}}{\partial \theta} - \psi(\epsilon\eta) \mathcal{U}_{k-2}(\eta, \theta, \xi), \\ f_k(0, \theta, \xi) &= \mathcal{P} f_k(0, \theta) + g_k(\theta, \xi) \text{ for } \sin(\theta + \xi) > 0, \\ \lim_{\eta \rightarrow \infty} f_k(\eta, \theta, \xi) &= f_k(\infty, \theta), \end{cases}$$

with

$$(5.32) \quad \mathcal{P} f_k(0, \theta) = 0.$$

where  $g_k = \vec{w} \cdot \nabla_x U_{k-1} - \mathcal{P}(\vec{w} \cdot \nabla_x U_{k-1}) + U_{k-2} - \mathcal{P} U_{k-2}$ . By the compatibility condition (5.99), we can define the  $(k-1)^{th}$  order interior solution as

$$(5.33) \quad \begin{cases} U_{k-1} &= \bar{U}_{k-1} - \vec{w} \cdot \nabla_x U_{k-2} - U_{k-3}, \\ \Delta_x \bar{U}_{k-1} - \bar{U}_{k-1} &= - \int_{S^1} (\vec{w} \cdot \nabla_x U_{k-2}) d\vec{w} - \int_{S^1} U_{k-3} d\vec{w} \text{ in } \Omega, \\ \frac{\partial \bar{U}_{k-1}}{\partial \vec{n}} &= \frac{1}{\pi} \int_{-\pi}^{\pi} \left( \vec{w} \cdot \nabla_x (\vec{w} \cdot U_{k-2} + U_{k-3}) \right) \sin \phi d\phi \\ &\quad + \frac{1}{\pi} \int_0^\infty \int_{-\pi}^\pi \left( \frac{\psi(\epsilon s)}{1 - \epsilon s} \cos(\theta + \xi) \frac{\partial \mathcal{U}_{k-1}}{\partial \theta} - \psi(\epsilon s) \mathcal{U}_{k-2} \right) (s, \theta, \xi) d\xi ds \text{ on } \partial\Omega. \end{cases}$$

In [4, pp.143], the author proved the expansion can be applied to the second order of  $\epsilon$ . Based on Remark 5.8 to the Milne problem, in order to show the existence of a solution  $f_2(\eta, \theta, \xi) \in L^\infty([0, \infty) \times [-\pi, \pi] \times [-\pi, \pi])$ , we at least require the source term

$$(5.34) \quad \cos(\theta + \xi) \frac{\psi}{1 - \epsilon\eta} \frac{\partial \mathcal{U}_1}{\partial \theta} \in L^\infty([0, \infty) \times [-\pi, \pi] \times [-\pi, \pi]),$$

since  $\mathcal{U}_0 = 0$ . We thus need to require

$$(5.35) \quad \frac{\partial(f_1 - f_1(\infty, \theta))}{\partial \theta} \in L^\infty([0, \infty) \times [-\pi, \pi] \times [-\pi, \pi]).$$

Since  $Z = \partial_\theta(f_1 - f_1(\infty))$  satisfies the equation

$$(5.36) \quad \begin{cases} \sin(\theta + \xi) \frac{\partial Z}{\partial \eta} + Z - \bar{Z} &= -\cos(\theta + \xi) \frac{\partial f_1}{\partial \eta}, \\ Z(0, \theta, \xi) &= \mathcal{P} Z(0, \theta) + \frac{\partial g_1}{\partial \theta}(\theta, \xi) \text{ for } \sin(\theta + \xi) > 0, \end{cases}$$

in order for  $Z \in L^\infty([0, \infty) \times [-\pi, \pi] \times [-\pi, \pi])$ , assuming the boundary data  $\partial_\theta g_1 \in L^\infty(\Gamma^-)$ , we require the source term

$$(5.37) \quad -\cos(\theta + \xi) \frac{\partial f_1}{\partial \eta} \in L^\infty([0, \infty) \times [-\pi, \pi] \times [-\pi, \pi]).$$

On the other hand, by Lemma B.1, we can show for specific  $g$ , it holds that  $\partial_\eta f_1 \notin L^\infty([0, \infty) \times [-\pi, \pi] \times [-\pi, \pi])$ . Due to the intrinsic singularity in the solution to (5.25), the construction in [4] breaks down.

**5.4.  $\epsilon$ -Milne Expansion with Geometric Correction.** In order to overcome the difficulty in estimating

$$(5.38) \quad \cos(\theta + \xi) \frac{\psi}{1 - \epsilon\eta} \frac{\partial \mathcal{U}_k}{\partial \theta},$$

we introduce one more substitution (1.40) to decompose this term and transform the equation (5.1) into

$$(5.39) \quad \begin{cases} \sin \phi \frac{\partial u^\epsilon}{\partial \eta} - \frac{\epsilon}{1 - \epsilon\eta} \cos \phi \left( \frac{\partial u^\epsilon}{\partial \phi} + \frac{\partial u^\epsilon}{\partial \theta} \right) + (1 + \epsilon^2) u^\epsilon - \frac{1}{2\pi} \int_{-\pi}^\pi u^\epsilon d\phi = 0, \\ u^\epsilon(0, \theta, \phi) = \mathcal{P} u^\epsilon(0, \theta) + \epsilon g(\theta, \phi) \text{ for } \sin \phi > 0, \end{cases}$$

with

$$(5.40) \quad \mathcal{P}u^\epsilon(0, \theta) = -\frac{1}{2} \int_{\sin \phi < 0} u^\epsilon(0, \theta, \xi) \sin \phi d\phi.$$

We define the  $\epsilon$ -Milne expansion of boundary layer as follows:

$$(5.41) \quad \mathcal{U}^\epsilon(\eta, \theta, \phi) \sim \sum_{k=0}^{\infty} \epsilon^k \mathcal{U}_k^\epsilon(\eta, \theta, \phi)$$

where  $\mathcal{U}_k^\epsilon$  can be defined by comparing the order of  $\epsilon$  via plugging (5.41) into the equation (5.39). Thus, in a neighborhood of the boundary, we have

$$(5.42) \quad \sin \phi \frac{\partial \mathcal{U}_0^\epsilon}{\partial \eta} + \frac{\epsilon}{1 - \epsilon \eta} \cos \phi \frac{\partial \mathcal{U}_0^\epsilon}{\partial \phi} + \mathcal{U}_0^\epsilon - \bar{\mathcal{U}}_0^\epsilon = 0$$

$$(5.43) \quad \sin \phi \frac{\partial \mathcal{U}_1^\epsilon}{\partial \eta} + \frac{\epsilon}{1 - \epsilon \eta} \cos \phi \frac{\partial \mathcal{U}_1^\epsilon}{\partial \phi} + \mathcal{U}_1^\epsilon - \bar{\mathcal{U}}_1^\epsilon = \frac{1}{1 - \epsilon \eta} \cos \phi \frac{\partial \mathcal{U}_0^\epsilon}{\partial \theta}$$

$$(5.44) \quad \sin \phi \frac{\partial \mathcal{U}_2^\epsilon}{\partial \eta} + \frac{\epsilon}{1 - \epsilon \eta} \cos \phi \frac{\partial \mathcal{U}_2^\epsilon}{\partial \phi} + \mathcal{U}_2^\epsilon - \bar{\mathcal{U}}_2^\epsilon = \frac{1}{1 - \epsilon \eta} \cos \phi \frac{\partial \mathcal{U}_1^\epsilon}{\partial \theta} - \mathcal{U}_0^\epsilon$$

$$(5.45) \quad \sin \phi \frac{\partial \mathcal{U}_k^\epsilon}{\partial \eta} + \frac{\epsilon}{1 - \epsilon \eta} \cos \phi \frac{\partial \mathcal{U}_k^\epsilon}{\partial \phi} + \mathcal{U}_k^\epsilon - \bar{\mathcal{U}}_k^\epsilon = \frac{1}{1 - \epsilon \eta} \cos \phi \frac{\partial \mathcal{U}_{k-1}^\epsilon}{\partial \theta} - \mathcal{U}_{k-2}^\epsilon$$

where

$$(5.46) \quad \bar{\mathcal{U}}_k^\epsilon(\eta, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathcal{U}_k^\epsilon(\eta, \theta, \phi) d\phi.$$

We refer to the cut-off function  $\psi$  and  $\psi_0$  as (1.26) and (1.27), and define the force as (1.48). Define the interior expansion as follows:

$$(5.47) \quad U^\epsilon(\vec{x}, \vec{w}) \sim \sum_{k=0}^{\infty} \epsilon^k U_k^\epsilon(\vec{x}, \vec{w})$$

where  $U_k^\epsilon$  satisfies the same equations as  $U_k$  in (5.12), (5.13) and (5.14). Here, to highlight its dependence on  $\epsilon$  via the  $\epsilon$ -Milne problem and boundary data, we add the superscript  $\epsilon$ .

The bridge between the interior solution and the boundary layer solution is the boundary condition of (5.1), so we first consider the boundary condition expansion:

$$(5.48) \quad (U_0^\epsilon + \mathcal{U}_0^\epsilon) = \mathcal{P}(U_0^\epsilon + \mathcal{U}_0^\epsilon),$$

$$(5.49) \quad (U_1^\epsilon + \mathcal{U}_1^\epsilon) = \mathcal{P}(U_1^\epsilon + \mathcal{U}_1^\epsilon) + g,$$

$$(5.50) \quad (U_2^\epsilon + \mathcal{U}_2^\epsilon) = \mathcal{P}(U_2^\epsilon + \mathcal{U}_2^\epsilon),$$

$$(5.51) \quad (U_k^\epsilon + \mathcal{U}_k^\epsilon) = \mathcal{P}(U_k^\epsilon + \mathcal{U}_k^\epsilon).$$

Note the fact that  $\bar{U}_k^\epsilon = \mathcal{P}\bar{U}_k^\epsilon$ , we can simplify above conditions as follows:

$$(5.52) \quad \mathcal{U}_0^\epsilon = \mathcal{P}\mathcal{U}_0^\epsilon,$$

$$(5.53) \quad \mathcal{U}_1^\epsilon = \mathcal{P}\mathcal{U}_1^\epsilon + (\vec{w} \cdot U_0^\epsilon - \mathcal{P}(\vec{w} \cdot U_0^\epsilon)) + g,$$

$$(5.54) \quad \mathcal{U}_2^\epsilon = \mathcal{P}\mathcal{U}_2^\epsilon + (\vec{w} \cdot U_1^\epsilon - \mathcal{P}(\vec{w} \cdot U_1^\epsilon)) + (U_0^\epsilon - \mathcal{P}U_0^\epsilon),$$

$$(5.55) \quad \mathcal{U}_k^\epsilon = \mathcal{P}\mathcal{U}_k^\epsilon + (\vec{w} \cdot U_{k-1}^\epsilon - \mathcal{P}(\vec{w} \cdot U_{k-1}^\epsilon)) + (U_{k-2}^\epsilon - \mathcal{P}U_{k-2}^\epsilon).$$

The construction of  $U_k^\epsilon$  and  $\mathcal{U}_k^\epsilon$  are as follows:

Step 1: Construction of  $\mathcal{U}_0^\epsilon$ .



Define the zeroth order boundary layer solution as

$$(5.56) \quad \begin{cases} \mathcal{U}_0^\epsilon(\eta, \theta, \phi) &= \psi_0(\epsilon\eta) \left( f_0^\epsilon(\eta, \theta, \phi) - f_0^\epsilon(\infty, \theta) \right), \\ \sin \phi \frac{\partial f_0^\epsilon}{\partial \eta} + F(\epsilon; \eta) \cos \phi \frac{\partial f_0^\epsilon}{\partial \phi} + f_0^\epsilon - \bar{f}_0^\epsilon &= 0, \\ f_0^\epsilon(0, \theta, \phi) &= \mathcal{P} f_0^\epsilon(0, \theta) \text{ for } \sin \phi > 0, \\ \lim_{\eta \rightarrow \infty} f_0^\epsilon(\eta, \theta, \phi) &= f_0^\epsilon(\infty, \theta), \end{cases}$$

with

$$(5.57) \quad \mathcal{P} f_0^\epsilon(0, \theta) = 0.$$

By Theorem 5.5,  $\mathcal{U}_0^\epsilon$  is well-defined. It is obvious to see  $f_0^\epsilon = f_0^\epsilon(\infty) = 0$  is the only solution.

Step 2: Construction of  $\mathcal{U}_1^\epsilon$  and  $U_0^\epsilon$ .

Define the first order boundary layer solution as

$$(5.58) \quad \begin{cases} \mathcal{U}_1^\epsilon(\eta, \theta, \phi) &= \psi_0(\epsilon\eta) \left( f_1^\epsilon(\eta, \theta, \phi) - f_1^\epsilon(\infty, \theta) \right), \\ \sin \phi \frac{\partial f_1^\epsilon}{\partial \eta} + F(\epsilon; \eta) \cos \phi \frac{\partial f_1^\epsilon}{\partial \phi} + f_1^\epsilon - \bar{f}_1^\epsilon &= \frac{\psi(\epsilon\eta)}{1 - \epsilon\eta} \cos \phi \frac{\partial \mathcal{U}_0^\epsilon}{\partial \theta}, \\ f_1^\epsilon(0, \theta, \phi) &= \mathcal{P} f_1^\epsilon(0, \theta) + g_1(\theta, \phi) \text{ for } \sin \phi > 0, \\ \lim_{\eta \rightarrow \infty} f_1^\epsilon(\eta, \theta, \phi) &= f_1^\epsilon(\infty, \theta), \end{cases}$$

with

$$(5.59) \quad \mathcal{P} f_1^\epsilon(0, \theta) = 0,$$

where

$$(5.60) \quad g_1 = (\vec{w} \cdot \nabla_x U_0^\epsilon(\vec{x}_0) - \mathcal{P}(\vec{w} \cdot \nabla_x U_0^\epsilon(\vec{x}_0))) + g,$$

with  $\vec{x}_0$  is the same boundary point as  $(0, \theta)$  and

$$(5.61) \quad \vec{w} = (-\sin(\phi - \theta), -\cos(\phi - \theta)),$$

$$(5.62) \quad \vec{n} = (\cos \theta, \sin \theta).$$

To solve (5.58), we require the compatibility condition (5.99) for the boundary data

$$(5.63) \quad \int_{\sin \phi > 0} \left( g + \vec{w} \cdot \nabla_x U_0^\epsilon(\vec{x}_0) - \mathcal{P}(\vec{w} \cdot \nabla_x U_0^\epsilon(\vec{x}_0)) \right) \sin \phi d\phi + \int_0^\infty \int_{-\pi}^\pi e^{-V(s)} \frac{\psi}{1 - \epsilon s} \cos \phi \frac{\partial \mathcal{U}_0^\epsilon}{\partial \theta}(s, \theta, \phi) d\phi ds = 0.$$

Note the fact

$$(5.64) \quad \begin{aligned} & \int_{\sin \phi > 0} \left( \vec{w} \cdot \nabla_x U_0^\epsilon(\vec{x}_0) - \mathcal{P}(\vec{w} \cdot \nabla_x U_0^\epsilon(\vec{x}_0)) \right) \sin \phi d\phi \\ &= \int_{\sin \phi > 0} (\vec{w} \cdot \nabla_x U_0^\epsilon(\vec{x}_0)) \sin \phi d\phi - 2\mathcal{P}(\vec{w} \cdot \nabla_x U_0^\epsilon(\vec{x}_0)) \\ &= \int_{\sin \phi > 0} (\vec{w} \cdot \nabla_x U_0^\epsilon(\vec{x}_0)) \sin \phi d\phi + \int_{\sin \phi < 0} (\vec{w} \cdot \nabla_x U_0^\epsilon(\vec{x}_0)) \sin \phi d\phi \\ &= \int_{-\pi}^\pi (\vec{w} \cdot \nabla_x U_0^\epsilon(\vec{x}_0)) \sin \phi d\phi \\ &= -\pi \nabla_x \bar{U}_0^\epsilon(\vec{x}_0) \cdot \vec{n} = -\pi \frac{\partial \bar{U}_0^\epsilon(\vec{x}_0)}{\partial \vec{n}}. \end{aligned}$$

We can simplify the compatibility condition as follows:

$$(5.65) \quad \int_{\sin \phi > 0} g(\phi) \sin \phi d\phi - \pi \frac{\partial \bar{U}_0^\epsilon(\vec{x}_0)}{\partial \vec{n}} + \int_0^\infty \int_{-\pi}^\pi e^{-V(s)} \frac{\psi}{1 - \epsilon s} \cos \phi \frac{\partial \mathcal{U}_0^\epsilon}{\partial \theta}(s, \theta, \phi) d\phi ds = 0.$$

Then we have

$$\begin{aligned}
 (5.66) \quad \frac{\partial \bar{U}_0^\epsilon(\vec{x}_0)}{\partial \vec{n}} &= \frac{1}{\pi} \int_{\sin \phi > 0} g(\theta, \phi) \sin \phi d\phi + \frac{1}{\pi} \int_0^\infty \int_{-\pi}^\pi e^{-V(s)} \frac{\psi}{1 - \epsilon s} \cos \phi \frac{\partial \mathcal{W}_0^\epsilon}{\partial \theta}(s, \theta, \phi) d\phi ds \\
 &= \frac{1}{\pi} \int_{\sin \phi > 0} g(\theta, \phi) \sin \phi d\phi.
 \end{aligned}$$

Hence, we define the zeroth order interior solution  $U_0^\epsilon(\vec{x}, \vec{w})$  as

$$(5.67) \quad \begin{cases} U_0^\epsilon &= \bar{U}_0^\epsilon, \\ \Delta_x \bar{U}_0^\epsilon - \bar{U}_0^\epsilon &= 0 \quad \text{in } \Omega, \\ \frac{\partial \bar{U}_0^\epsilon}{\partial \vec{n}} &= \frac{1}{\pi} \int_{\sin \phi > 0} g(\theta, \phi) \sin \phi d\phi \quad \text{on } \partial\Omega. \end{cases}$$

Step 3: Analysis of  $U_0^\epsilon$  and  $\mathcal{W}_1^\epsilon$ .

By Theorem 5.6, we can easily see  $f_1^\epsilon$  is well-defined in  $L^\infty(\Omega \times \mathcal{S}^1)$  and approaches  $f_1^\epsilon(\infty)$  exponentially fast as  $\eta \rightarrow \infty$ . Since  $f_0^\epsilon = 0$ , then if  $g \in C^r(\Gamma^-)$ , it is obvious to check  $\partial_n U_0^\epsilon \in C^r(\partial\Omega)$ . Hence, by the standard elliptic estimate, there exists a unique solution  $U_0^\epsilon \in W^{r+1,p}(\Omega)$  for arbitrary  $p \geq 2$  satisfying

$$(5.68) \quad \|\bar{U}_0^\epsilon\|_{W^{r+1,p}(\Omega)} \leq C(\Omega) \left\| \frac{\partial \bar{U}_0^\epsilon}{\partial \vec{n}} \right\|_{W^{r-1/p,p}(\partial\Omega)},$$

which implies  $\nabla_x U_0^\epsilon \in W^{r,p}(\Omega)$ ,  $\nabla_x U_0^\epsilon \in W^{r-1/p,p}(\partial\Omega)$  and  $U_0^\epsilon \in C^{r,1-2/p}(\Omega)$ .

Step 4: Construction of  $\mathcal{W}_2^\epsilon$  and  $U_1^\epsilon$ .

Define the second order boundary layer solution as

$$(5.69) \quad \begin{cases} \mathcal{W}_2^\epsilon(\eta, \theta, \phi) &= \psi(\epsilon\eta) \left( f_2^\epsilon(\eta, \theta, \phi) - f_2^\epsilon(\infty, \theta) \right), \\ \sin \phi \frac{\partial f_2^\epsilon}{\partial \eta} + F(\epsilon; \eta) \cos \phi \frac{\partial f_2^\epsilon}{\partial \phi} + f_2^\epsilon - \bar{f}_2^\epsilon &= \frac{\psi(\epsilon\eta)}{1 - \epsilon\eta} \cos \phi \frac{\partial \mathcal{W}_1^\epsilon}{\partial \theta} - \psi(\epsilon\eta) \mathcal{W}_0^\epsilon, \\ f_2^\epsilon(0, \theta, \phi) &= \mathcal{P} f_2^\epsilon(0, \theta) + g_2(\theta, \phi) \quad \text{for } \sin \phi > 0, \\ \lim_{\eta \rightarrow \infty} f_2^\epsilon(\eta, \theta, \phi) &= f_2^\epsilon(\infty, \theta), \end{cases}$$

with

$$(5.70) \quad \mathcal{P} f_2^\epsilon(0, \theta) = 0,$$

where

$$(5.71) \quad g_2 = (\vec{w} \cdot \nabla_x U_1^\epsilon(\vec{x}_0) - \mathcal{P}(\vec{w} \cdot \nabla_x U_1^\epsilon(\vec{x}_0))) + U_0^\epsilon(\vec{x}_0) - \mathcal{P} U_0^\epsilon(\vec{x}_0).$$

In order for equation (5.69) being well-posed, we require the compatibility condition (5.99) for the boundary data

$$\begin{aligned}
 (5.72) \quad & \int_{\sin \phi > 0} \left( \vec{w} \cdot \nabla_x U_1^\epsilon(\vec{x}_0) - \mathcal{P}(\vec{w} \cdot \nabla_x U_1^\epsilon(\vec{x}_0)) \right) \sin \phi d\phi \\
 & + \int_0^\infty \int_{-\pi}^\pi e^{-V(s)} \left( \frac{\psi}{1 - \epsilon s} \cos \phi \frac{\partial \mathcal{W}_1^\epsilon}{\partial \theta}(s, \theta, \phi) - \psi \mathcal{W}_0^\epsilon(s, \theta, \phi) \right) d\phi ds = 0.
 \end{aligned}$$

Similarly, we can directly verify the relation

$$(5.73) \quad \int_{\sin \phi > 0} \left( \vec{w} \cdot \nabla_x U_1^\epsilon(\vec{x}_0) - \mathcal{P}(\vec{w} \cdot \nabla_x U_1^\epsilon(\vec{x}_0)) \right) \sin \phi d\phi = -\pi \frac{\partial \bar{U}_1^\epsilon(\vec{x}_0)}{\partial \vec{n}}.$$

We can simplify the compatibility condition as follows:

$$(5.74) \quad -\pi \frac{\partial \bar{U}_1^\epsilon(\vec{x}_0)}{\partial \vec{n}} + \int_0^\infty \int_{-\pi}^\pi e^{-V(s)} \left( \frac{\psi}{1 - \epsilon s} \cos \phi \frac{\partial \mathcal{W}_1^\epsilon}{\partial \theta}(s, \theta, \phi) - \psi \mathcal{W}_0^\epsilon(s, \theta, \phi) \right) d\phi ds = 0.$$

Then we have

$$\begin{aligned}
 (5.75) \quad -\pi \frac{\partial \bar{U}_1^\epsilon(\vec{x}_0)}{\partial \vec{n}} &= \frac{1}{\pi} \int_0^\infty \int_{-\pi}^\pi e^{-V(s)} \left( \frac{\psi}{1-\epsilon s} \cos \phi \frac{\partial \mathcal{U}_1^\epsilon}{\partial \theta}(s, \theta, \phi) - \psi \mathcal{U}_0^\epsilon(s, \theta, \phi) \right) d\phi ds \\
 &= \frac{1}{\pi} \int_0^\infty \int_{-\pi}^\pi e^{-V(s)} \frac{\psi}{1-\epsilon s} \cos \phi \frac{\partial \mathcal{U}_1^\epsilon}{\partial \theta}(s, \theta, \phi) d\phi ds.
 \end{aligned}$$

Hence, we define the first order interior solution  $U_1^\epsilon(\vec{x})$  as

$$(5.76) \quad \begin{cases} U_1^\epsilon &= \bar{U}_1^\epsilon - \vec{w} \cdot \nabla_x U_0^\epsilon, \\ \Delta_x \bar{U}_1^\epsilon - \bar{U}_1^\epsilon &= - \int_{S^1} (\vec{w} \cdot \nabla_x U_0^\epsilon) d\vec{w} \text{ in } \Omega, \\ \frac{\partial \bar{U}_1^\epsilon}{\partial \vec{n}} &= \frac{1}{\pi} \int_0^\infty \int_{-\pi}^\pi e^{-V(s)} \frac{\psi(\epsilon s)}{1-\epsilon s} \cos \phi \frac{\partial \mathcal{U}_1^\epsilon}{\partial \theta}(s, \theta, \phi) d\phi ds \text{ on } \partial\Omega. \end{cases}$$

Step 5: Analysis of  $U_1^\epsilon$  and  $\mathcal{U}_2^\epsilon$ .

By Theorem 5.6, we can easily see  $f_2^\epsilon$  is well-defined in  $L^\infty(\Omega \times \mathcal{S}^1)$  and approaches  $f_2^\epsilon(\infty)$  exponentially fast as  $\eta \rightarrow \infty$ . By above analysis, it is obvious to check  $\partial_n \bar{U}_1^\epsilon \in C^{r-1}(\partial\Omega)$ . Hence, by the standard elliptic estimate, there exists a unique solution  $\bar{U}_1^\epsilon \in W^{r,p}(\Omega)$  for arbitrary  $p \geq 2$  satisfying

$$(5.77) \quad \|\bar{U}_1^\epsilon\|_{W^{r,p}(\Omega)} \leq C(\Omega) \left( \left\| \frac{\partial \bar{U}_1^\epsilon}{\partial \vec{n}} \right\|_{W^{r-1-1/p,p}(\partial\Omega)} + \|\nabla_x U_0^\epsilon\|_{W^{r-2,p}(\Omega)} \right)$$

which implies  $\nabla_x U_1^\epsilon \in W^{r-1,p}(\Omega)$ ,  $\nabla_x U_1^\epsilon \in W^{r-1-1/p,p}(\partial\Omega)$  and  $U_1^\epsilon \in C^{r-1,1-2/p}(\Omega)$ .

Step 6: Generalization to arbitrary  $k$ .

Then this process can proceed to arbitrary  $k$  as long as  $g$  is sufficiently smooth. We can always determine  $\mathcal{U}_k^\epsilon$  and  $U_{k-1}^\epsilon$  simultaneously based on the compatibility condition. Define the  $k^{th}$  order boundary layer solution as

$$(5.78) \quad \begin{cases} \mathcal{U}_k^\epsilon(\eta, \theta, \phi) &= \psi(\epsilon\eta) \left( f_k^\epsilon(\eta, \theta, \phi) - f_k^\epsilon(\infty, \theta) \right), \\ \sin \phi \frac{\partial f_k^\epsilon}{\partial \eta} + F(\epsilon; \eta) \cos \phi \frac{\partial f_k^\epsilon}{\partial \phi} + f_k^\epsilon - \bar{f}_k^\epsilon &= \frac{\psi}{1-\epsilon\eta} \cos \phi \frac{\partial \mathcal{U}_{k-1}^\epsilon}{\partial \theta} - \psi \mathcal{U}_{k-2}^\epsilon, \\ f_k^\epsilon(0, \theta, \phi) &= \mathcal{P} f_k^\epsilon(0, \theta) + g_k(\theta, \phi) \text{ for } \sin \phi > 0, \\ \lim_{\eta \rightarrow \infty} f_k^\epsilon(\eta, \theta, \phi) &= f_k^\epsilon(\infty, \theta), \end{cases}$$

with

$$(5.79) \quad \mathcal{P} f_k^\epsilon(0, \theta) = 0,$$

where

$$(5.80) \quad g_k = (\vec{w} \cdot \nabla_x U_{k-1}^\epsilon(\vec{x}_0) - \mathcal{P}(\vec{w} \cdot \nabla_x U_{k-1}^\epsilon(\vec{x}_0))) + U_{k-2}^\epsilon(\vec{x}_0) - \mathcal{P} U_{k-2}^\epsilon(\vec{x}_0).$$

Hence, we define the  $(k-1)^{th}$  order interior solution as

$$(5.81) \quad \begin{cases} U_{k-1}^\epsilon &= \bar{U}_{k-1}^\epsilon - \vec{w} \cdot \nabla_x U_{k-2}^\epsilon - U_{k-3}^\epsilon, \\ \Delta_x \bar{U}_{k-1}^\epsilon - \bar{U}_{k-1}^\epsilon &= - \int_{S^1} (\vec{w} \cdot \nabla_x U_{k-2}^\epsilon) d\vec{w} - \int_{S^1} U_{k-3}^\epsilon d\vec{w} \text{ in } \Omega, \\ \frac{\partial \bar{U}_{k-1}^\epsilon}{\partial \vec{n}} &= \frac{1}{\pi} \int_{-\pi}^\pi \left( \vec{w} \cdot \nabla_x (\vec{w} \cdot U_{k-2}^\epsilon + U_{k-3}^\epsilon) \right) \sin \phi d\phi \\ &\quad + \frac{1}{\pi} \int_0^\infty \int_{-\pi}^\pi e^{-V(s)} \left( \frac{\psi(\epsilon s)}{1-\epsilon s} \cos \phi \frac{\partial \mathcal{U}_{k-1}^\epsilon}{\partial \theta} - \psi(\epsilon s) \mathcal{U}_{k-2}^\epsilon \right) (s, \theta, \phi) d\phi ds \text{ on } \partial\Omega. \end{cases}$$

In particular, for  $g \in C^{k+1}(\Gamma^-)$ , we can define the interior solution up to  $k^{th}$  order and the boundary layer solution up to  $(k+1)^{th}$  order, i.e. up to  $U_k^\epsilon$  and  $\mathcal{U}_{k+1}^\epsilon$ .

**5.5. Well-Posedness of Steady Neutron Transport Equation.** In this section, we consider the well-posedness of the steady neutron transport equation

$$(5.82) \quad \begin{cases} \epsilon \vec{w} \cdot \nabla_x u + (1 + \epsilon^2)u - \bar{u} &= f(\vec{x}, \vec{w}) \text{ in } \Omega, \\ u(\vec{x}_0, \vec{w}) &= \mathcal{P}u(\vec{x}_0) + \epsilon g(\vec{x}_0, \vec{w}) \text{ for } \vec{w} \cdot \vec{n} < 0 \text{ and } \vec{x}_0 \in \partial\Omega. \end{cases}$$

We define the  $L^\infty$  norms in  $\Omega \times \mathcal{S}^1$  as usual:

$$(5.83) \quad \|f\|_{L^\infty(\Omega \times \mathcal{S}^1)} = \sup_{(\vec{x}, \vec{w}) \in \Omega \times \mathcal{S}^1} |f(\vec{x}, \vec{w})|.$$

**Theorem 5.1.** *Assume  $f(\vec{x}, \vec{w}) \in L^\infty(\Omega \times \mathcal{S}^1)$  and  $g(x_0, \vec{w}) \in L^\infty(\Gamma^-)$ . Then for the transport equation*

$$(5.84) \quad \begin{cases} \epsilon \vec{w} \cdot \nabla_x u + (1 + \epsilon^2)u - \bar{u} &= f(\vec{x}, \vec{w}) \text{ in } \Omega, \\ u(\vec{x}_0, \vec{w}) &= \mathcal{P}u(\vec{x}_0) + \epsilon g(\vec{x}_0, \vec{w}) \text{ for } \vec{x}_0 \in \partial\Omega \text{ and } \vec{w} \cdot \vec{n} < 0, \end{cases}$$

*there exists a unique solution  $u(\vec{x}, \vec{w}) \in L^\infty(\Omega \times \mathcal{S}^1)$  satisfying*

$$(5.85) \quad \|u\|_{L^\infty(\Omega \times \mathcal{S}^1)} \leq C \left( \frac{1}{\epsilon^2} \|f\|_{L^\infty(\Omega \times \mathcal{S}^1)} + \frac{1}{\epsilon^2} \|g\|_{L^\infty(\Gamma^-)} \right).$$

*Proof.* We iteratively construct an approximating sequence  $\{u^k\}_{k=0}^\infty$  where  $u^0 = 0$  and

$$(5.86) \quad \begin{cases} \epsilon \vec{w} \cdot \nabla_x u^k + (1 + \epsilon^2)u^k - \bar{u}^{k-1} &= f(\vec{x}, \vec{w}) \text{ in } \Omega, \\ u^k(\vec{x}_0, \vec{w}) &= \mathcal{P}u^{k-1}(\vec{x}_0) + \epsilon g(\vec{x}_0, \vec{w}) \text{ for } \vec{x}_0 \in \partial\Omega \text{ and } \vec{w} \cdot \vec{n} < 0. \end{cases}$$

By Lemma 2.1, this sequence is well-defined and  $\|u^k\|_{L^\infty(\Omega \times \mathcal{S}^1)} < \infty$ . We rewrite equation (5.86) along the characteristics as

$$(5.87) \quad u^k(\vec{x}, \vec{w}) = (\epsilon g + \mathcal{P}u^{k-1})(\vec{x} - \epsilon t_b \vec{w}, \vec{w}) e^{-(1+\epsilon^2)t_b} + \int_0^{t_b} (f + \bar{u}^{k-1})(\vec{x} - \epsilon(t_b - s)\vec{w}, \vec{w}) e^{-(1+\epsilon^2)(t_b-s)} ds.$$

where the backward exit time  $t_b$  is defined as (2.13). We define the difference  $v^k = u^k - u^{k-1}$  for  $k \geq 1$ . Then  $v^k$  satisfies

$$(5.88) \quad v^{k+1}(\vec{x}, \vec{w}) = \mathcal{P}v^k(\vec{x} - \epsilon t_b \vec{w}, \vec{w}) e^{-(1+\epsilon^2)t_b} + \int_0^{t_b} \bar{v}^k(\vec{x} - \epsilon(t_b - s)\vec{w}, \vec{w}) e^{-(1+\epsilon^2)(t_b-s)} ds.$$

Since  $\|\bar{v}^k\|_{L^\infty(\Omega \times \mathcal{S}^1)} \leq \|v^k\|_{L^\infty(\Omega \times \mathcal{S}^1)}$  and  $\|\mathcal{P}v^k\|_{L^\infty(\Omega \times \mathcal{S}^1)} \leq \|v^k\|_{L^\infty(\Omega \times \mathcal{S}^1)}$ , we can directly estimate

$$(5.89) \quad \begin{aligned} \|v^{k+1}\|_{L^\infty(\Omega \times \mathcal{S}^1)} &\leq e^{-(1+\epsilon^2)t_b} \|v^{k+1}\|_{L^\infty(\Omega \times \mathcal{S}^1)} + \|v^k\|_{L^\infty(\Omega \times \mathcal{S}^1)} \int_0^{t_b} e^{-(1+\epsilon^2)(t_b-s)} ds \\ &\leq e^{-(1+\epsilon^2)t_b} \|v^{k+1}\|_{L^\infty(\Omega \times \mathcal{S}^1)} + \frac{1}{1+\epsilon^2} (1 - e^{-(1+\epsilon^2)t_b}) \|v^k\|_{L^\infty(\Omega \times \mathcal{S}^1)}. \end{aligned}$$

Hence, we naturally have

$$(5.90) \quad \|v^{k+1}\|_{L^\infty(\Omega \times \mathcal{S}^1)} \leq \frac{1}{1+\epsilon^2} \|v^k\|_{L^\infty(\Omega \times \mathcal{S}^1)}.$$

Thus, this is a contraction iteration. Considering  $v^1 = u^1$ , we have

$$(5.91) \quad \|v^k\|_{L^\infty(\Omega \times \mathcal{S}^1)} \leq \left( \frac{1}{1+\epsilon^2} \right)^{k-1} \|u^1\|_{L^\infty(\Omega \times \mathcal{S}^1)}.$$

for  $k \geq 1$ . Therefore,  $u^k$  converges strongly in  $L^\infty$  to the limiting solution  $u$  satisfying

$$(5.92) \quad \|u\|_{L^\infty(\Omega \times \mathcal{S}^1)} \leq \sum_{k=1}^\infty \|v^k\|_{L^\infty(\Omega \times \mathcal{S}^1)} \leq \frac{1+\epsilon^2}{\epsilon^2} \|u^1\|_{L^\infty(\Omega \times \mathcal{S}^1)} \leq \frac{2}{\epsilon^2} \|u^1\|_{L^\infty(\Omega \times \mathcal{S}^1)}.$$

Since  $u^1$  satisfies the equation

$$u^1(\vec{x}, \vec{w}) = g(\vec{x} - \epsilon t_b \vec{w}, \vec{w}) e^{-(1+\epsilon^2)t_b} + \int_0^{t_b} f(\vec{x} - \epsilon(t_b - s)\vec{w}, \vec{w}) e^{-(1+\epsilon^2)(t_b-s)} ds.$$

Based on Lemma 2.1, we can directly estimate

$$(5.93) \quad \|u^1\|_{L^\infty(\Omega \times \mathcal{S}^1)} \leq \|f\|_{L^\infty(\Omega \times \mathcal{S}^1)} + \|g\|_{L^\infty(\Gamma^-)}.$$

Combining (5.92) and (5.93), we can naturally obtain the existence and estimates. Also, the uniqueness easily follows from the energy estimates.  $\square$

**Theorem 5.2.** *Assume  $g(x_0, \vec{w}) \in L^\infty(\Gamma^-)$ . Then for the steady neutron transport equation (5.1), there exists a unique solution  $u^\epsilon(\vec{x}, \vec{w}) \in L^\infty(\Omega \times \mathcal{S}^1)$  satisfying*

$$(5.94) \quad \|u^\epsilon\|_{L^\infty(\Omega \times \mathcal{S}^1)} \leq \frac{C}{\epsilon^2} \|g\|_{L^\infty(\Gamma^-)}.$$

*Proof.* We can apply Theorem 5.1 to the equation (5.1). The result naturally follows.  $\square$

**5.6.  $\epsilon$ -Milne Problem.** We consider the  $\epsilon$ -Milne problem for  $f^\epsilon(\eta, \theta, \phi)$  in the domain  $(\eta, \theta, \phi) \in [0, \infty) \times [-\pi, \pi) \times [-\pi, \pi)$

$$(5.95) \quad \begin{cases} \sin \phi \frac{\partial f^\epsilon}{\partial \eta} + F(\epsilon; \eta) \cos \phi \frac{\partial f^\epsilon}{\partial \phi} + f^\epsilon - \bar{f}^\epsilon &= S^\epsilon(\eta, \theta, \phi), \\ f^\epsilon(0, \theta, \phi) &= h^\epsilon(\theta, \phi) + \mathcal{P}f^\epsilon(0) \text{ for } \sin \phi > 0, \\ \lim_{\eta \rightarrow \infty} f^\epsilon(\eta, \theta, \phi) &= f_\infty^\epsilon(\theta), \end{cases}$$

where

$$(5.96) \quad \mathcal{P}f^\epsilon(\eta, \theta) = -\frac{1}{2} \int_{\sin \phi < 0} f^\epsilon(\eta, \theta, \phi) \sin \phi d\phi,$$

$F(\epsilon; \eta)$  is defined as (1.48),

$$(5.97) \quad |h^\epsilon(\theta, \phi)| \leq M,$$

and

$$(5.98) \quad |S^\epsilon(\eta, \theta, \phi)| \leq M e^{-K\eta},$$

for  $M$  and  $K$  uniform in  $\epsilon$  and  $\theta$ .

For notational simplicity, we omit  $\epsilon$  and  $\theta$  dependence in  $f^\epsilon$  in this section. The same convention also applies to  $F(\epsilon; \eta)$ ,  $V(\epsilon; \eta)$ ,  $S^\epsilon(\eta, \theta, \phi)$  and  $h^\epsilon(\theta, \phi)$ . However, our estimates are independent of  $\epsilon$  and  $\theta$ .

**5.6.1. Compatibility Condition.**

**Lemma 5.3.** *In order for the equation (5.95) to have a solution  $f \in L^\infty([0, \infty) \times [-\pi, \pi))$ , the boundary data  $h$  and the source term  $S$  must satisfy the compatibility condition*

$$(5.99) \quad \int_{\sin \phi > 0} h(\phi) \sin \phi d\phi + \int_0^\infty \int_{-\pi}^\pi e^{-V(s)} S(s, \phi) d\phi ds = 0.$$

*In particular, if  $S = 0$ , then the compatibility condition reduces to*

$$(5.100) \quad \int_{\sin \phi > 0} h(\phi) \sin \phi d\phi = 0.$$

*Proof.* We just integrate over  $\phi \in [-\pi, \pi)$  on both sides of (5.95) and integrate by parts to achieve

$$(5.101) \quad \frac{d}{d\eta} \int_{-\pi}^\pi f(\eta, \phi) \sin \phi d\phi + F(\eta) \int_{-\pi}^\pi f(\eta, \phi) \sin \phi d\phi = \int_{-\pi}^\pi S(\eta, \phi) d\phi.$$

This is a first order ordinary differential equation for  $\int_{-\pi}^\pi f(\eta, \phi) \sin \phi d\phi$ . The initial data is

$$(5.102) \quad \begin{aligned} \int_{-\pi}^\pi f(0, \phi) \sin \phi d\phi &= \int_{\sin \phi > 0} f(0, \phi) \sin \phi d\phi + \int_{\sin \phi < 0} f(0, \phi) \sin \phi d\phi \\ &= \int_{\sin \phi > 0} \left( h(\phi) + \mathcal{P}f(0) \right) \sin \phi d\phi + \int_{\sin \phi < 0} f(0, \phi) \sin \phi d\phi \\ &= \int_{\sin \phi > 0} h(\phi) \sin \phi d\phi + \int_{\sin \phi > 0} \mathcal{P}f(0) \sin \phi d\phi - 2\mathcal{P}f(0) \\ &= \int_{\sin \phi > 0} h(\phi) \sin \phi d\phi. \end{aligned}$$

Hence, we can directly solve (5.101) as

$$(5.103) \quad \int_{-\pi}^{\pi} f(\eta, \phi) \sin \phi d\phi = e^{V(\eta)} \left( \int_{\sin \phi > 0} h(\phi) \sin \phi d\phi + \int_0^{\eta} \int_{-\pi}^{\pi} e^{-V(s)} S(s, \phi) d\phi ds \right).$$

The problem (5.95) requires  $f(\eta, \phi) \rightarrow f_{\infty}$  as  $\eta \rightarrow \infty$ . Hence, we must have

$$(5.104) \quad \lim_{\eta \rightarrow \infty} \int_{-\pi}^{\pi} f(\eta, \phi) \sin \phi d\phi = 0.$$

Therefore, the only possibility to justify above requirement is

$$(5.105) \quad \int_{\sin \phi > 0} h(\phi) \sin \phi d\phi + \int_0^{\infty} \int_{-\pi}^{\pi} e^{-V(s)} S(s, \phi) d\phi ds = 0.$$

This is the desired compatibility condition. If  $S = 0$ , then above condition reduces to

$$(5.106) \quad \int_{\sin \phi > 0} h(\phi) \sin \phi d\phi = 0.$$

□

**5.6.2. Reduction to In-flow  $\epsilon$ -Milne Problem.** It is easy to see if  $f$  is a solution to (5.95), then  $f + C$  is also a solution for any constant  $C$ . Hence, in order to obtain a unique solution, we need a normalization condition

$$(5.107) \quad \mathcal{P}f(0) = 0.$$

The following lemma tells us the problem (5.95) can be reduced to the  $\epsilon$ -Milne problem (3.1) with in-flow boundary.

**Lemma 5.4.** *If the boundary data  $h$  and  $S$  satisfy the compatibility condition (5.99), then the solution  $f$  to the  $\epsilon$ -Milne problem (3.1) with in-flow boundary as  $f = h$  on  $\sin \phi > 0$  is also a solution to the  $\epsilon$ -Milne problem (5.95) with diffusive boundary, which satisfies the normalization condition (5.107). Furthermore, this is the unique solution to (5.95) among the functions satisfying (5.107) and  $\|f - f_{\infty}\|_{L^2 L^2} < \infty$ .*

*Proof.* Consider  $f$  satisfies the  $\epsilon$ -Milne problem with in-flow boundary as follows:

$$(5.108) \quad \begin{cases} \sin \phi \frac{\partial f}{\partial \eta} + F(\eta) \cos \phi \frac{\partial f}{\partial \phi} + f - \bar{f} = S(\eta, \phi), \\ f(0, \phi) = h(\phi) \text{ for } \sin \phi > 0, \\ \lim_{\eta \rightarrow \infty} f(\eta, \phi) = f_{\infty}. \end{cases}$$

Then there exists a  $f_{\infty}$  such that  $\|f - f_{\infty}\|_{L^{\infty} L^{\infty}} < \infty$  and  $f$  decays to  $f_{\infty}$  exponentially. Therefore,  $z = f - f_{\infty}$  satisfies the equation

$$(5.109) \quad \begin{cases} \sin \phi \frac{\partial z}{\partial \eta} + F(\eta) \cos \phi \frac{\partial z}{\partial \phi} + z - \bar{z} = S(\eta, \phi), \\ z(0, \phi) = h(\phi) - f_{\infty} \text{ for } \sin \phi > 0, \\ \lim_{\eta \rightarrow \infty} z(\eta, \phi) = 0. \end{cases}$$

Multiplying  $e^{-V(\eta)}$  on both sides of (5.109) and integrating over  $\phi \in [-\pi, \pi]$  imply

$$(5.110) \quad \frac{d}{d\eta} \int_{-\pi}^{\pi} e^{-V(\eta)} z(\eta, \phi) \sin \phi d\phi = \int_{-\pi}^{\pi} e^{-V(\eta)} S(\eta, \phi) d\phi.$$

Since  $z$  decays exponentially with respect to  $\eta$ , we know  $z \in L^1([0, \infty) \times [-\pi, \pi])$ . Hence, we can integrate (5.110) over  $\eta \in [0, \infty)$  to obtain

$$(5.111) \quad 0 - \int_{-\pi}^{\pi} e^{-V(0)} z(0, \phi) \sin \phi d\phi = \int_0^{\infty} \int_{-\pi}^{\pi} e^{-V(s)} S(s, \phi) d\phi ds,$$

which further implies

$$(5.112) \quad \int_{-\pi}^{\pi} z(0, \phi) \sin \phi d\phi + \int_0^{\infty} \int_{-\pi}^{\pi} e^{-V(s)} S(s, \phi) d\phi ds = 0.$$

Then we may compute

$$\begin{aligned}
 (5.113) \quad \int_{-\pi}^{\pi} z(0, \phi) \sin \phi d\phi &= \int_{\sin \phi > 0} z(0, \phi) \sin \phi d\phi + \int_{\sin \phi < 0} z(0, \phi) \sin \phi d\phi \\
 &= \int_{\sin \phi > 0} (h(\phi) - f_{\infty}) \sin \phi d\phi - 2\mathcal{P}z(0) \\
 &= \int_{\sin \phi > 0} h(\phi) \sin \phi d\phi - 2\left(\mathcal{P}z(0) + f_{\infty}\right).
 \end{aligned}$$

Combining (5.112), (5.113) and the compatibility condition (5.99), we have

$$(5.114) \quad \mathcal{P}z(0) + f_{\infty} = 0.$$

Since

$$(5.115) \quad \mathcal{P}z(0) = \mathcal{P}f(0) - \mathcal{P}f_{\infty}(0) = \mathcal{P}f(0) - f_{\infty}.$$

naturally, we have  $\mathcal{P}f(0) = 0$ . Hence,  $f$  is a solution to the  $\epsilon$ -Milne problem (5.95) with the normalization condition (5.107). By Cauchy's inequality, we can deduce the fact that  $\|f - f_{\infty}\|_{L^2 L^2} < \infty$  implies  $\|\bar{f} - f_{\infty}\|_{L^2 L^2} < \infty$ . Then by a similar argument as Step 3 in the proof of Lemma 3.4, we can show the uniqueness.  $\square$

In summary, based on above analysis, we can utilize the known result for  $\epsilon$ -Milne problem with in-flow boundary to obtain the well-posedness, decay and maximum principle of the solution to the  $\epsilon$ -Milne problem (5.95).

**Theorem 5.5.** *There exists a unique solution  $f(\eta, \phi)$  to the  $\epsilon$ -Milne problem (5.95) with the normalization condition (5.107) satisfying*

$$(5.116) \quad \|f - f_{\infty}\|_{L^{\infty} L^{\infty}} \leq C \left(1 + M + \frac{M}{K}\right).$$

**Theorem 5.6.** *For  $K_0 > 0$  sufficiently small, the solution  $f(\eta, \phi)$  to the  $\epsilon$ -Milne problem (5.95) with the normalization condition (5.107) satisfies*

$$(5.117) \quad \|e^{K_0 \eta}(f - f_{\infty})\|_{L^{\infty} L^{\infty}} \leq C \left(1 + M + \frac{M}{K}\right).$$

**Theorem 5.7.** *The solution to the  $\epsilon$ -Milne problem (5.95) with  $S = 0$  and the normalization condition (5.107) satisfies the maximum principle, i.e.*

$$(5.118) \quad \min_{\sin \phi > 0} h(\phi) \leq f(\eta, \phi) \leq \max_{\sin \phi > 0} h(\phi).$$

**Remark 5.8.** *Note that when  $F = 0$ , then all the previous proofs can be recovered and Theorem 5.5, Theorem 5.6 and Theorem 5.7 still hold. Hence, we can obtain the well-posedness, decay and maximum principle of the classical Milne problem*

$$(5.119) \quad \begin{cases} \sin \phi \frac{\partial f}{\partial \eta} + f - \bar{f} = S(\eta, \phi), \\ f(0, \phi) = \mathcal{P}f(0) + h(\phi) \text{ for } \sin \phi > 0, \\ \lim_{\eta \rightarrow \infty} f(\eta, \phi) = f_{\infty}, \end{cases}$$

with the normalization condition (5.107). Note that now we always have  $V(\eta) = 0$ .

## 5.7. Main Results.

**Theorem 5.9.** *Assume  $g(\vec{x}_0, \vec{w}) \in C^3(\Gamma^-)$ . Then for the steady neutron transport equation (5.1), the unique solution  $u^{\epsilon}(\vec{x}, \vec{w}) \in L^{\infty}(\Omega \times \mathcal{S}^1)$  satisfies*

$$(5.120) \quad \|u^{\epsilon} - U_0^{\epsilon} - \mathcal{U}_0^{\epsilon}\|_{L^{\infty}} = O(\epsilon),$$

Moreover, if  $g(\theta, \phi) = \cos \phi$ , then there exists a  $C > 0$  such that

$$(5.121) \quad \|u^{\epsilon} - U_0 - \mathcal{U}_0 - \epsilon U_1 - \epsilon \mathcal{U}_1\|_{L^{\infty}} \geq C\epsilon > 0,$$

when  $\epsilon$  is sufficiently small.

*Proof.* We can divide the proof into several steps:

Step 1: Remainder definitions.

Note that the boundary layer solution depends on  $\epsilon$  due to the force and the interior solution also depends on  $\epsilon$  due the boundary condition. We may rewrite the asymptotic expansion as follows:

$$(5.122) \quad u^\epsilon \sim \sum_{k=0}^{\infty} \epsilon^k U_k^\epsilon + \sum_{k=0}^{\infty} \epsilon^k \mathcal{U}_k^\epsilon.$$

The remainder can be defined as

$$(5.123) \quad R_N = u^\epsilon - \sum_{k=0}^N \epsilon^k U_k^\epsilon - \sum_{k=0}^N \epsilon^k \mathcal{U}_k^\epsilon = u - Q_N - \mathcal{Q}_N,$$

where

$$(5.124) \quad Q_N = \sum_{k=0}^N \epsilon^k U_k^\epsilon,$$

$$(5.125) \quad \mathcal{Q}_N = \sum_{k=0}^N \epsilon^k \mathcal{U}_k^\epsilon.$$

Noting the equation (5.39) is equivalent to the equation (5.1), we write  $\mathcal{L}$  to denote the neutron transport operator as follows:

$$(5.126) \quad \begin{aligned} \mathcal{L}u &= \epsilon \vec{w} \cdot \nabla_x u + (1 + \epsilon^2)u - \bar{u} \\ &= \sin \phi \frac{\partial u}{\partial \eta} - \frac{\epsilon}{1 - \epsilon \eta} \cos \phi \left( \frac{\partial u}{\partial \phi} + \frac{\partial u}{\partial \theta} \right) + (1 + \epsilon^2)u - \bar{u} \end{aligned}$$

Step 2: Estimates of  $\mathcal{L}Q_N$ .

The interior contribution can be estimated as

$$(5.127) \quad \mathcal{L}Q_0 = \epsilon \vec{w} \cdot \nabla_x Q_0 + (1 + \epsilon^2)Q_0 - \bar{Q}_0 = (Q_0 - \bar{Q}_0) + \epsilon \vec{w} \cdot \nabla_x U_0^\epsilon + \epsilon^2 U_0^\epsilon = \epsilon \vec{w} \cdot \nabla_x U_0^\epsilon + \epsilon^2 U_0^\epsilon.$$

We can directly estimate

$$(5.128) \quad |\epsilon \vec{w} \cdot \nabla_x U_0^\epsilon| \leq C\epsilon |\nabla_x U_0^\epsilon| \leq C\epsilon,$$

$$(5.129) \quad |\epsilon^2 U_0^\epsilon| \leq C\epsilon^2 |U_0^\epsilon| \leq C\epsilon^2.$$

This implies

$$(5.130) \quad |\mathcal{L}Q_0| \leq C\epsilon.$$

For higher order term, we can estimate

$$(5.131) \quad \mathcal{L}Q_N = \epsilon \vec{w} \cdot \nabla_x Q_N + (1 + \epsilon^2)Q_N - \bar{Q}_N = \epsilon^{N+1} \vec{w} \cdot \nabla_x U_N^\epsilon + \epsilon^{N+2} U_N^\epsilon + \epsilon^{N+1} U_{N-1}^\epsilon.$$

We have

$$(5.132) \quad |\epsilon^{N+1} \vec{w} \cdot \nabla_x U_N^\epsilon| \leq C\epsilon^{N+1} |\nabla_x U_N^\epsilon| \leq C\epsilon^{N+1},$$

$$(5.133) \quad |\epsilon^{N+2} U_N^\epsilon + \epsilon^{N+1} U_{N-1}^\epsilon| \leq C\epsilon^{N+2} |U_N^\epsilon| + C\epsilon^{N+1} |U_{N-1}^\epsilon| \leq C\epsilon^{N+1}.$$

This implies

$$(5.134) \quad |\mathcal{L}Q_N| \leq C\epsilon^{N+1}.$$

Step 3: Estimates of  $\mathcal{L}\mathcal{Q}_N$ .



The boundary layer solution is  $\mathcal{U}_k^\epsilon = (f_k^\epsilon - f_k^\epsilon(\infty)) \cdot \psi_0 = \mathcal{V}_k \psi_0$  where  $f_k^\epsilon(\eta, \theta, \phi)$  solves the  $\epsilon$ -Milne problem and  $\mathcal{V}_k = f_k^\epsilon - f_k^\epsilon(\infty)$ . Notice  $\psi_0 \psi = \psi_0$ , so the boundary layer contribution can be estimated as

(5.135)

$$\begin{aligned}
\mathcal{L}\mathcal{Q}_0 &= \sin \phi \frac{\partial \mathcal{Q}_0}{\partial \eta} - \frac{\epsilon}{1-\epsilon\eta} \cos \phi \left( \frac{\partial \mathcal{Q}_0}{\partial \phi} + \frac{\partial \mathcal{Q}_0}{\partial \theta} \right) + (1+\epsilon^2)\mathcal{Q}_0 - \bar{\mathcal{Q}}_0 \\
&= \sin \phi \left( \psi_0 \frac{\partial \mathcal{V}_0}{\partial \eta} + \mathcal{V}_0 \frac{\partial \psi_0}{\partial \eta} \right) - \frac{\psi_0 \epsilon}{1-\epsilon\eta} \cos \phi \left( \frac{\partial \mathcal{V}_0}{\partial \phi} + \frac{\partial \mathcal{V}_0}{\partial \theta} \right) + (1+\epsilon^2)\psi_0 \mathcal{V}_0 - \psi_0 \bar{\mathcal{V}}_0 \\
&= \sin \phi \left( \psi_0 \frac{\partial \mathcal{V}_0}{\partial \eta} + \mathcal{V}_0 \frac{\partial \psi_0}{\partial \eta} \right) - \frac{\psi_0 \psi \epsilon}{1-\epsilon\eta} \cos \phi \left( \frac{\partial \mathcal{V}_0}{\partial \phi} + \frac{\partial \mathcal{V}_0}{\partial \theta} \right) + (1+\epsilon^2)\psi_0 \mathcal{V}_0 - \psi_0 \bar{\mathcal{V}}_0 \\
&= \psi_0 \left( \sin \phi \frac{\partial \mathcal{V}_0}{\partial \eta} - \frac{\epsilon \psi}{1-\epsilon\eta} \cos \phi \frac{\partial \mathcal{V}_0}{\partial \phi} + \mathcal{V}_0 - \bar{\mathcal{V}}_0 \right) + \sin \phi \frac{\partial \psi_0}{\partial \eta} \mathcal{V}_0 - \frac{\psi_0 \epsilon}{1-\epsilon\eta} \cos \phi \frac{\partial \mathcal{V}_0}{\partial \theta} + \epsilon^2 \psi_0 \mathcal{V}_0 \\
&= \sin \phi \frac{\partial \psi_0}{\partial \eta} \mathcal{V}_0 - \frac{\psi_0 \epsilon}{1-\epsilon\eta} \cos \phi \frac{\partial \mathcal{V}_0}{\partial \theta} + \epsilon^2 \psi_0 \mathcal{V}_0.
\end{aligned}$$

Since  $\psi_0 = 1$  when  $\eta \leq 1/(4\epsilon)$ , the effective region of  $\partial_\eta \psi_0$  is  $\eta \geq 1/(4\epsilon)$  which is further and further from the origin as  $\epsilon \rightarrow 0$ . By Theorem 5.6, the first term in (5.135) can be controlled as

$$(5.136) \quad \left| \sin \phi \frac{\partial \psi_0}{\partial \eta} \mathcal{V}_0 \right| \leq C e^{-\frac{\kappa_0}{\epsilon}} \leq C\epsilon.$$

For the second term in (5.135), we have

$$(5.137) \quad \left| -\frac{\psi_0 \epsilon}{1-\epsilon\eta} \cos \phi \frac{\partial \mathcal{V}_0}{\partial \theta} \right| \leq C\epsilon \left| \frac{\partial \mathcal{V}_0}{\partial \theta} \right| \leq C\epsilon.$$

For the third term in (5.135), we have

$$(5.138) \quad |\epsilon^2 \psi_0 \mathcal{V}_0| \leq C\epsilon.$$

This implies

$$(5.139) \quad |\mathcal{L}\mathcal{Q}_0| \leq C\epsilon.$$

For higher order term, we can estimate

$$\begin{aligned}
(5.140) \quad \mathcal{L}\mathcal{Q}_N &= \sin \phi \frac{\partial \mathcal{Q}_N}{\partial \eta} - \frac{\epsilon}{1-\epsilon\eta} \cos \phi \left( \frac{\partial \mathcal{Q}_N}{\partial \phi} + \frac{\partial \mathcal{Q}_N}{\partial \theta} \right) + (1+\epsilon^2)\mathcal{Q}_N - \bar{\mathcal{Q}}_N \\
&= \sum_{i=0}^N \epsilon^i \sin \phi \frac{\partial \psi_0}{\partial \eta} \mathcal{V}_i - \frac{\psi_0 \epsilon^{N+1}}{1-\epsilon\eta} \cos \phi \frac{\partial \mathcal{V}_N}{\partial \theta} + \epsilon^{N+2} \psi_0 \mathcal{V}_N + \epsilon^{N+1} \psi_0 \mathcal{V}_{N-1}.
\end{aligned}$$

Away from the origin, the first term in (5.140) can be controlled as

$$(5.141) \quad \left| \sum_{i=0}^N \epsilon^i \sin \phi \frac{\partial \psi_0}{\partial \eta} \mathcal{V}_i \right| \leq C e^{-\frac{\kappa_0}{\epsilon}} \leq C\epsilon^{N+1}.$$

For the second term in (5.140), we have

$$(5.142) \quad \left| -\frac{\psi_0 \epsilon^{N+1}}{1-\epsilon\eta} \cos \phi \frac{\partial \mathcal{V}_N}{\partial \theta} \right| \leq C\epsilon^{N+1} \left| \frac{\partial \mathcal{V}_N}{\partial \theta} \right| \leq C\epsilon^{N+1}.$$

For the third term in (5.140), we have

$$(5.143) \quad |\epsilon^{N+2} \psi_0 \mathcal{V}_N + \epsilon^{N+1} \psi_0 \mathcal{V}_{N-1}| \leq C\epsilon^{N+1}.$$

This implies

$$(5.144) \quad |\mathcal{L}\mathcal{Q}_N| \leq C\epsilon^{N+1}.$$

Step 4: Proof of (5.120).

In summary, since  $\mathcal{L}u^\epsilon = 0$ , collecting (5.123), (5.134) and (5.144), we can prove

$$(5.145) \quad |\mathcal{L}R_N| \leq C\epsilon^{N+1}.$$

Consider the asymptotic expansion to  $N = 2$ , then the remainder  $R_2$  satisfies the equation

$$(5.146) \quad \begin{cases} \epsilon \vec{w} \cdot \nabla_x R_2 + R_2 - \bar{R}_2 &= \mathcal{L}R_2 \text{ for } \vec{x} \in \Omega, \\ R_2 &= \mathcal{P}R_2 \text{ for } \vec{w} \cdot \vec{n} < 0 \text{ and } \vec{x}_0 \in \partial\Omega. \end{cases}$$

By Theorem 5.1, we have

$$(5.147) \quad \|R_2\|_{L^\infty(\Omega \times \mathcal{S}^1)} \leq \frac{C}{\epsilon^2} \|\mathcal{L}R_2\|_{L^\infty(\Omega \times \mathcal{S}^1)} \leq \frac{C\epsilon^3}{\epsilon^2} \leq C\epsilon.$$

Hence, we have

$$(5.148) \quad \left\| u^\epsilon - \sum_{k=0}^2 \epsilon^k U_k^\epsilon - \sum_{k=0}^2 \epsilon^k \mathcal{U}_k^\epsilon \right\|_{L^\infty} = O(\epsilon).$$

Since it is easy to see

$$(5.149) \quad \left\| \sum_{k=1}^2 \epsilon^k U_k^\epsilon + \sum_{k=1}^2 \epsilon^k \mathcal{U}_k^\epsilon \right\|_{L^\infty} \leq C\epsilon,$$

our result naturally follows. This completes the proof of (5.120).

Step 5: Proof of (5.121).

By (5.25), the solution  $f_1$  satisfies the Milne problem

$$(5.150) \quad \begin{cases} \sin(\theta + \xi) \frac{\partial f_1}{\partial \eta} + f_1 - \bar{f}_1 &= 0, \\ f_1(0, \theta, \xi) &= \mathcal{P}f_1(0, \theta) + g_1(\theta, \xi) \text{ for } \sin(\theta + \xi) > 0, \\ \lim_{\eta \rightarrow \infty} f_1(\eta, \theta, \xi) &= f_1(\infty, \theta). \end{cases}$$

For convenience of comparison, we make the substitution  $\phi = \theta + \xi$  to obtain

$$(5.151) \quad \begin{cases} \sin \phi \frac{\partial f_1}{\partial \eta} + f_1 - \bar{f}_1 &= 0, \\ f_1(0, \phi) &= \mathcal{P}f_1(0) + g_1(\theta, \phi) \text{ for } \sin \phi > 0, \\ \lim_{\eta \rightarrow \infty} f_1(\eta, \theta, \xi) &= f_1(\infty, \theta). \end{cases}$$

Assume (5.121) is incorrect. For our  $g(\phi) = \cos \phi$  which is independent of  $\theta$ , since  $\mathcal{U}_0 = \mathcal{U}_0^\epsilon = 0$  and  $U_0 = U_0^\epsilon = 0$ , we have

$$(5.152) \quad \lim_{\epsilon \rightarrow 0} \|(U_1 + \mathcal{U}_1) - (U_1^\epsilon + \mathcal{U}_1^\epsilon)\|_{L^\infty} = 0.$$

Since now  $\mathcal{U}_1$  and  $\mathcal{U}_1^\epsilon$  are independent of  $\theta$ , by (5.30) and (5.76), we can directly estimate

$$(5.153) \quad \frac{\partial \bar{U}_1^\epsilon}{\partial \vec{n}} = - \int_0^\infty \int_{-\pi}^\pi e^{-V(s)} \left( \frac{\psi(\epsilon s)}{1 - \epsilon \eta} \cos \phi \frac{\partial \mathcal{U}_1^\epsilon}{\partial \theta}(s, \phi) - \psi(\epsilon s) \mathcal{U}_0^\epsilon(s, \phi) \right) d\phi ds = 0,$$

and also

$$(5.154) \quad \frac{\partial \bar{U}_1}{\partial \vec{n}} = - \int_0^\infty \int_{-\pi}^\pi e^{-V(s)} \left( \frac{\psi(\epsilon s)}{1 - \epsilon \eta} \cos \phi \frac{\partial \mathcal{U}_1}{\partial \theta}(s, \phi) - \psi(\epsilon s) \mathcal{U}_0(s, \phi) \right) d\phi ds = 0.$$

Hence, we have  $\bar{U}_1^\epsilon = \bar{U}_1$  in the domain, which further implies  $U_1^\epsilon = U_1$ . Therefore, we can obtain

$$(5.155) \quad \lim_{\epsilon \rightarrow 0} \|\mathcal{U}_1 - \mathcal{U}_1^\epsilon\|_{L^\infty} = 0.$$

Then on the boundary of  $\sin \phi > 0$ , these two boundary layer solutions satisfy

$$(5.156) \quad \mathcal{U}_1 = g - f_1(\infty),$$

$$(5.157) \quad \mathcal{U}_1^\epsilon = g - f_1^\epsilon(\infty).$$

Naturally, we have the estimate  $\lim_{\epsilon \rightarrow 0} \|f_1^\epsilon(\infty) - f_1(\infty)\|_{L^\infty} = 0$  based on above assumptions. Hence, we may further derive

$$(5.158) \quad \lim_{\epsilon \rightarrow \infty} \|(f_1(\infty) + \mathcal{U}_1) - (f_1^\epsilon(\infty) + \mathcal{U}_1^\epsilon)\|_{L^\infty} = 0.$$

For  $0 \leq \eta \leq 1/(2\epsilon)$ , we have  $\psi_0 = 1$ , which means  $f_1 = \mathcal{U}_1 + f_1(\infty)$  and  $f_1^\epsilon = \mathcal{U}_1^\epsilon + f_1^\epsilon(\infty)$  on  $[0, 1/(2\epsilon)]$ . Since we have  $\mathcal{P}\mathcal{U}_1^\epsilon(0) = -f_1^\epsilon(\infty)$  and  $\mathcal{P}\mathcal{U}_1(0) = -f_1(\infty)$ , we have recovered the normalization condition,

i.e.  $\mathcal{P}f_1(0) = \mathcal{P}f_1^\epsilon(0) = 0$ . Note that  $g$  satisfies the compatibility condition (5.100). Therefore, the  $\epsilon$ -Milne problem satisfied by  $f_1$  and  $f_1^\epsilon$  can be reduced to the  $\epsilon$ -Milne problem with in-flow boundary. Hence, we can naturally obtain the desired result through the proof of Theorem 1.2.  $\square$

#### APPENDIX A. CONSTRUCTION OF THE COUNTEREXAMPLE WITH IN-FLOW BOUNDARY

**Lemma A.1.** *For the Milne problem*

$$(A.1) \quad \begin{cases} \sin(\theta + \xi) \frac{\partial f}{\partial \eta} + f - \bar{f} &= 0, \\ f(0, \theta, \xi) &= g(\theta, \xi) \text{ for } \sin(\theta + \xi) > 0, \\ \lim_{\eta \rightarrow \infty} f(\eta, \theta, \xi) &= f(\infty, \theta), \end{cases}$$

if  $g(\theta, \xi) = \cos(3(\theta + \xi))$ , then we have

$$(A.2) \quad \frac{\partial f}{\partial \eta} \notin L^\infty([0, \infty) \times [-\pi, \pi] \times [-\pi, \pi]).$$

*Proof.* We divide the proof into several steps: we first assume  $\partial_\eta f \in L^\infty([0, \infty) \times [-\pi, \pi] \times [-\pi, \pi])$  and then show it can lead to a contradiction.

Step 1: Maximum principle

Theorem 3.14 implies the solution  $f$  to the problem (A.1) satisfies the maximum principle, i.e. for any  $(\eta, \theta, \phi)$

$$(A.3) \quad \min_{\sin(\theta+\xi)>0} g(\theta, \xi) \leq f(\eta, \theta, \xi) \leq \max_{\sin(\theta+\xi)>0} g(\theta, \xi).$$

We can see the data  $g(\theta, \xi) = \cos(3(\theta + \xi))$  satisfying  $g(\theta, -\theta) = 1$  and  $|g| \leq 1$ . Based on the maximum principle, we can derive for any  $(\eta, \theta, \xi)$

$$(A.4) \quad f(\eta, \theta, \xi) \leq 1.$$

Hence, certainly we have  $f(0, \theta, \xi) \leq 1$  for  $\sin(\theta + \xi) < 0$ .

Step 2: Estimates of  $\bar{f}(0, \theta)$ .

We can directly estimate

$$(A.5) \quad \begin{aligned} \bar{f}(0, \theta) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(0, \theta, \xi) d\xi \\ &= \frac{1}{2\pi} \left( \int_{\sin(\theta+\xi)<0} f(0, \theta, \xi) d\xi + \int_{\sin(\theta+\xi)>0} f(0, \theta, \xi) d\xi \right) \\ &\leq \frac{1}{2\pi} \int_{\sin(\theta+\phi)>0} f(0, \theta, \xi) d\xi + \frac{1}{2}. \end{aligned}$$

By the choice of  $g$ , we naturally have

$$(A.6) \quad \int_{\sin(\theta+\xi)>0} f(0, \theta, \xi) d\xi = \int_{\sin(\theta+\xi)>0} g(\theta, \xi) d\xi = 0.$$

Then this implies

$$(A.7) \quad \bar{f}(0, \theta) \leq \frac{1}{2}.$$

Step 3: Definition of trace.

It is easy to see  $\partial_\eta f$  satisfies the Milne problem

$$(A.8) \quad \sin(\theta + \xi) \frac{\partial(\partial_\eta f)}{\partial \eta} + \partial_\eta f - \overline{\partial_\eta f} = 0.$$

Since we have  $\partial_\eta f \in L^\infty([0, \infty) \times [-\pi, \pi] \times [-\pi, \pi])$  which implies  $\overline{\partial_\eta f} \in L^\infty([0, L] \times [-\pi, \pi])$ , by Ukai's trace theorem, we may define the trace of  $\partial_\eta f$  on  $\eta = 0$  satisfying  $\partial_\eta f(0, \theta, \phi) \in L^\infty[-\pi, \pi] \times [-\pi, \pi]$ .

However, we can define the trace of  $\partial_\eta f$  in another fashion. For any  $\xi \neq -\theta$  and  $\xi \neq \pi - \theta$ , we have  $\sin(\theta + \xi) \neq 0$ . Since we have  $f \in L^\infty([0, \infty) \times [-\pi, \pi) \times [-\pi, \pi))$  as well as  $\bar{f} \in L^\infty[0, \infty) \times [-\pi, \pi)$ , by the Milne problem (A.1), it is naturally to define for  $\eta > 0$

$$(A.9) \quad \partial_\eta f(\eta, \theta, \xi) = \frac{\bar{f}(\eta, \theta) - f(\eta, \theta, \xi)}{\sin(\theta + \xi)}.$$

Since  $\partial_\eta f \in L^\infty([0, \infty) \times [-\pi, \pi) \times [-\pi, \pi))$ , we know  $f$  is continuous with respect to  $\eta$  for a.e.  $(\theta, \xi)$ . Taking  $\eta \rightarrow 0$  defines the trace for  $\partial_\eta f$  at  $(0, \theta, \xi)$

$$(A.10) \quad \partial_\eta f(0, \theta, \xi) = \frac{\bar{f}(0, \theta) - f(0, \theta, \xi)}{\sin(\theta + \xi)}.$$

Since the grazing set  $\{(\theta, \xi) : \theta + \xi = 0 \text{ or } \theta + \xi = \pi\}$  is zero-measured on the boundary  $\eta = 0$ , then we have the trace of  $\partial_\eta f$  is a.e. well-defined.

By the uniqueness of trace of  $\partial_\eta f$ , above two types of traces must coincide with each other a.e.. Then we may combine them both and obtain  $\partial_\eta f(0, \theta, \xi) \in L^\infty[-\pi, \pi) \times [-\pi, \pi)$  is a.e. well-defined and satisfies the formula

$$(A.11) \quad \partial_\eta f(0, \theta, \xi) = \frac{\bar{f}(0, \theta) - f(0, \theta, \xi)}{\sin(\theta + \xi)}.$$

Step 4: Contradiction.

Therefore, we may consider the limiting process

$$(A.12) \quad \lim_{\xi \rightarrow -\theta^+} \frac{\partial f}{\partial \eta}(0, \theta, \xi) = \lim_{\xi \rightarrow -\theta^+} \frac{\bar{f}(0, \theta) - f(0, \theta, \xi)}{\sin(\theta + \xi)}.$$

Since we know as  $\xi \rightarrow -\theta^+$ , it follows that

$$(A.13) \quad \sin(\theta + \xi) \rightarrow 0^+$$

$$(A.14) \quad \bar{f}(0, \theta) - f(0, \theta, \xi) \rightarrow \bar{f}(0, \theta) - g(\theta, -\theta) = \bar{f}(0, \theta) - 1 < 0.$$

Then this leads to

$$(A.15) \quad \lim_{\xi \rightarrow -\theta^+} \frac{\partial f}{\partial \eta}(0, \theta, \xi) = -\infty.$$

which means  $\partial_\eta f(0, \theta, \xi) \notin L^\infty[-\pi, \pi) \times [-\pi, \pi)$ . This contradicts our result in the previous step. Hence, our assumption that  $\partial_\eta f \in L^\infty([0, \infty) \times [-\pi, \pi) \times [-\pi, \pi))$  cannot be true.

Step 5: Another contradiction.

There is another way to show this fact. Since  $\partial_\eta f \in L^\infty([0, L] \times [-\pi, \pi) \times [-\pi, \pi))$ . Also, we have  $f \in L^\infty([0, L] \times [-\pi, \pi) \times [-\pi, \pi))$ . Then this implies  $f$  is Lipschitz continuous in  $[0, \infty)$  with respect to  $\eta$  for a.e.  $(\theta, \xi)$ . Hence, this implies  $\bar{f}$  is also Lipschitz continuous in  $\eta \in [0, \infty)$ . Without loss of generality, we may assume  $\|\partial_\eta f\|_\infty \leq M$ . Thus we have for a.e.  $(\theta, \xi) \in \sin(\theta + \phi) > 0$

$$(A.16) \quad |f(\eta, \theta, \xi) - f(0, \theta, \xi)| \leq M\eta$$

$$(A.17) \quad |\bar{f}(\eta, \theta) - \bar{f}(0, \theta)| \leq M\eta.$$

Then

$$(A.18) \quad \begin{aligned} |f(\eta, \theta, \xi) - \bar{f}(\eta, \theta)| &\geq |f(0, \theta, \xi) - \bar{f}(0, \theta)| - |f(\eta, \theta, \xi) - f(0, \theta, \xi)| - |\bar{f}(\eta, \theta) - \bar{f}(0, \theta)| \\ &\geq |\bar{f}(0, \theta) - g(\theta, -\theta)| - 2M\eta. \end{aligned}$$

Since we know  $|\bar{f}(0, \theta) - g(\theta, -\theta)| \geq C > 0$  for some constant  $C$ . Then as long as  $\eta \leq C/(4M)$ , we have

$$(A.19) \quad |f(\eta, \theta, \xi) - \bar{f}(\eta, \theta)| \geq \frac{C}{2}.$$

Since for  $(\theta, \xi)$  not in the grazing set, we always have

$$(A.20) \quad \partial_\eta f(\eta, \theta, \xi) = \frac{\bar{f}(\eta, \theta) - f(\eta, \theta, \xi)}{\sin(\theta + \xi)}.$$

then  $|\partial_\eta f|$  can be arbitrarily large as long as  $\sin(\theta + \xi)$  is sufficiently small, and also it possesses a positive measure. This implies  $\partial_\eta f \in L^\infty([0, \infty) \times [-\pi, \pi) \times [-\pi, \pi))$  cannot be true.  $\square$

## APPENDIX B. CONSTRUCTION OF THE COUNTEREXAMPLE WITH DIFFUSIVE BOUNDARY

**Lemma B.1.** *For the Milne problem*

$$(A.1) \quad \begin{cases} \sin(\theta + \xi) \frac{\partial f}{\partial \eta} + f - \bar{f} = 0, \\ f(0, \theta, \xi) = \mathcal{P}f(0, \theta) + g_1(\theta, \xi) \text{ for } \sin(\theta + \xi) > 0, \\ \lim_{\eta \rightarrow \infty} f(\eta, \theta, \xi) = f(\infty, \theta), \end{cases}$$

with

$$(A.2) \quad \mathcal{P}f(0, \theta) = 0.$$

If  $g(\theta, \xi) = \cos(3(\theta + \xi))$ , then we have

$$(A.3) \quad \frac{\partial f}{\partial \eta} \notin L^\infty([0, \infty) \times [-\pi, \pi) \times [-\pi, \pi)).$$

*Proof.* For  $g(\theta, \xi) = \cos(3(\theta + \xi))$ , we can easily derive  $U_0 = 0$ . Hence  $g_1 = g$ . Note that  $g$  satisfies the compatibility condition (5.100). With the normalization condition  $\mathcal{P}f(0, \theta) = 0$ , we can see this problem reduces to the Milne problem with in-flow boundary

$$(A.4) \quad \begin{cases} \sin(\theta + \xi) \frac{\partial f}{\partial \eta} + f - \bar{f} = 0, \\ f(0, \theta, \xi) = g(\theta, \xi) \text{ for } \sin(\theta + \xi) > 0, \\ \lim_{\eta \rightarrow \infty} f(\eta, \theta, \xi) = f_1(\infty, \theta). \end{cases}$$

Therefore, we can complete the proof by Theorem A.1.  $\square$

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